A zero-test for D-algebraic transseries

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The zero-test problem

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- $f_1, \ldots, f_k$: computable series over $\mathbb{K}$

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A series f *is said to be* ***D-algebraic*** *if there exists a non-zero differential polynomial* $P \in \mathbb{K}[\delta^{\mathbb{N}} F]$ *with* $P(f) = 0$.

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A series f is said to be ***D-algebraic*** *if there exists a non-zero differential polynomial $P \in \mathbb{K}[\delta^{\mathbb{N}} F]$ with $P(f) = 0$.*

Main focus: f_i is a grid-based D-algebraic transseries, represented by (P_i, f_i).

Transseries: a first glimpse

A ***transseries*** generalizes $\mathbb{R}[[z]]$ by closing it under exp, log and infinite summation.

$$x^{-1} + 2\,x^{-2} + 3\,x^{-3} + \cdots$$

Here regard x as a positive infinite indeterminate.

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- forms a field closed under differentiation and composition;
- includes the resolution of many differential equations;
- effective computation, analogous to multivariate Laurent series.

Transseries and transbasis

- $\mathfrak{B}$: a transbasis $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$

A grid-based transseries can be represented as

$$f = \sum_{\substack{\alpha_1, \ldots, \alpha_n \in \mathbb{R} \\ (\alpha_1, \ldots, \alpha_n) \geqslant (\nu_1, \ldots, \nu_n)}} f_{\alpha_1, \ldots, \alpha_n} \cdot \mathfrak{b}_1^{-\alpha_1} \cdots \mathfrak{b}_n^{-\alpha_n}.$$

- $v(f) = \min \{(\alpha_1, \ldots, \alpha_n) \mid f_{\alpha_1, \ldots, \alpha_n} \neq 0\}$
- $f \prec g$: if $v(f) > v(g)$
- $\mathbb{B} = \mathbb{R}[[\mathfrak{b}_1^{-\mathbb{R}}; \ldots; \mathfrak{b}_n^{-\mathbb{R}}]]$: the field of all grid-based transseries w.r.t. $\mathfrak{B}$
- $\delta_1, \ldots, \delta_n$: natural derivations on $\mathbb{R}[[\mathfrak{b}_1^{-\mathbb{R}}; \ldots; \mathfrak{b}_n^{-\mathbb{R}}]]$

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Example: With $\mathfrak{B} = (\log x, x, x^x, \mathrm{e}^{x^\pi + x})$, we can write

$$\begin{aligned} f &= (\log x)^\pi + x^{-1} + 2x^{-2} + 6x^{-3} + 24x^{-4} + \cdots + x^{-x}\mathrm{e}^{-(x^\pi + x)} + x^{-2x}\mathrm{e}^{-2(x^\pi + x)} + \cdots \\ &= \mathfrak{b}_1^\pi + \mathfrak{b}_2^{-1} + 2\mathfrak{b}_2^{-2} + 6\mathfrak{b}_2^{-3} + 24\mathfrak{b}_2^{-4} + \cdots + \mathfrak{b}_3^{-1}\mathfrak{b}_4^{-1} + \mathfrak{b}_3^{-2}\mathfrak{b}_4^{-2} + \cdots. \end{aligned}$$

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Therefore, $v(f) = (-\pi, 0, 0, 0)$ and $v_4(f) = 0$.

Ritt reduction

- $\mathbb{B}\{F\} = \mathbb{B}[\delta_1^{\mathbb{N}} F]$: ring of (univariate) differential polynomials over $\mathbb{B}$
- P: a non-trivial differential polynomial in $\mathbb{B}\{F\} \setminus \mathbb{B}$
- ℓ_P: the highest $\delta_1^i F$ occuring in P, called the **leader** of P
- $\operatorname{rank} P := (\ell_P, \deg_{\ell_P} P)$, called the **Ritt rank** of P

Reducibility

$P, Q_1, \ldots, Q_l \in \mathbb{B}\{F\} \setminus \mathbb{B}$

$$P \text{ is } \textbf{\textit{reducible}} \text{ w.r.t. } Q_1, \ldots, Q_l \iff \exists i,\ \operatorname{rank} P \geqslant \operatorname{rank} Q_i$$

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- I_P: the leading coefficient of P, called the **initial** of P
- $S_P := \partial P / \partial \ell_P$, called the **separant** of P

Ritt reduction

$\mathbb{N}$ $\qquad$ $\mathbb{B}\{F\}[\delta_1]$ $\qquad$ reduced w.r.t. $Q_1, \ldots, Q_l$

$$I_{Q_1}^{\alpha_1} \cdots I_{Q_l}^{\alpha_l} S_{Q_1}^{\beta_1} \cdots S_{Q_l}^{\beta_l} P \;=\; \Theta_1 Q_1 + \cdots + \Theta_l Q_l + R$$

$$P \operatorname{rem} Q \;:=\; P \operatorname{rem} (Q_1, \ldots, Q_l) \;:=\; R$$

Valuation extension

$v(f) \in \mathbb{R}^n \cup \{\infty\}$: valuation in $(\mathfrak{b}_1^{-1}; \ldots; \mathfrak{b}_n^{-1})$ of $f \in \mathbb{B} = \mathbb{R}[[\mathfrak{b}_1^{-1}; \ldots; \mathfrak{b}_n^{-1}]]$

Valuation extends to $\mathbb{R}[[\mathfrak{b}_1^{-1}; \ldots; \mathfrak{b}_n^{-1}]]\{F\} \subseteq \mathbb{R}\{F\}[[\mathfrak{b}_1^{-1}; \ldots; \mathfrak{b}_n^{-1}]]$

- P: a differential polynomial written as $P = \sum_{\boldsymbol{i} = (i_0, \ldots, i_r)} P_{\boldsymbol{i}} \prod_j (\delta_1^j F)^{i_j}$
- $P_{[d]} := \sum_{|\boldsymbol{i}| = d} P_{\boldsymbol{i}} \delta_1^{\boldsymbol{i}} F$, where $|\boldsymbol{i}| := \sum_j i_j$ and $\delta_1^{\boldsymbol{i}} F := \prod_j (\delta_1^j F)^{i_j}$

Definition

The following equation is said to be ***quasi-linear*** *if* $v(P) = v(P_{[1]}) < v(P_{[0]})$.

$$P(f) = 0, \quad f \prec 1$$

Quasi-linear equations

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Example: With $\mathfrak{B} = (x, \mathrm{e}^x)$, the equation $P(f) = 0$, $f \prec 1$ is quasi-linear, where

$$P := \frac{x-1}{\mathrm{e}^{2x}} + \left(\frac{1}{x} - \frac{1}{\mathrm{e}^x} + \frac{1}{x\,\mathrm{e}^x}\right) \delta_1 F + \left(1 - \frac{x}{\mathrm{e}^x}\right) F + \frac{F \delta_1 F}{x} + F^2 \in \mathbb{B}[\delta_1^{\mathbb{N}} F].$$

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Example: With $\mathfrak{B}=(x,\mathrm{e}^x)$, the equation $P(f)=0$, $f\prec 1$ is quasi-linear, where

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Because $v(P)=v(P_{[1]})=(0,0)<v(P_{[0]})=(-1,2)$.

Indicial polynomial $\mathrm{I}_P \in \mathbb{R}[N]$ of homogeneous $P \in \mathbb{B}\{F\}$ of degree d

Writing $P = \sum_{|\boldsymbol{i}|=d} P_{\boldsymbol{i}} \delta_n^{\boldsymbol{i}} F \in \mathbb{B}[\delta_n^{\mathbb{N}} F]$,

$$\mathrm{I}_P(N) := \sum_{\boldsymbol{i}} (P_{\boldsymbol{i}})_{v(P)} (-N)^{\|\boldsymbol{i}\|} \in \mathbb{R}[N], \qquad \|\boldsymbol{i}\| := i_1 + 2\, i_2 + \cdots + r\, i_r$$

For general $P \in \mathbb{B}\{F\}$ and $f \in \mathbb{B}$ s.t. $P(f) = 0$ and $f \prec 1$, we mainly focus on $\mathrm{I}_{P,f} := \mathrm{I}_{P_{+f,[1]}}$.

- P_{+f}: the differential polynomial s.t. $P_{+f}(\varepsilon) = P(f + \varepsilon)$ for any $\varepsilon \in \mathbb{B}$

$$Z_{P,f} := \begin{cases} \infty, & \text{if } \mathrm{I}_{P,f} = 0 \\ v_n(f), & \text{if } \mathrm{I}_{P,f}(\lambda) \neq 0 \text{ for all } \lambda \in \mathbb{R} \\ \max\{\lambda \in \mathbb{R} : \mathrm{I}_{P,f}(\lambda) = 0\}, & \text{otherwise} \end{cases}$$

Note: $\mathrm{I}_{P,f} \neq 0$ whenever $P(f) = 0$, $f \prec 1$ is quasi-linear.

Algorithms for power series

- R. H. Risch. Algebraic properties of elementary functions in analysis. *Amer. Journ. of Math.*, 4(101):743–759, 1975.
- J. Denef and L. Lipshitz. Power series solutions of algebraic differential equations. *Math. Ann.*, 267:213–238, 1984.
- J. Shackell. A differential-equations approach to functional equivalence. In *Proc. ISSAC'89*, pages 7–10. Portland, Oregon, ACM, New York, 1989. ACM Press.
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Reduction and generalization

Let $(P_1, f_1), \ldots, (P_k, f_k) \in \mathbb{R}[\delta_1^{\mathbb{N}} F] \times \mathbb{B}$ s.t. $P_i(f_i) = 0$.

Given $Q \in \mathbb{R}[\delta_1^{\mathbb{N}} F_1, \ldots, \delta_1^{\mathbb{N}} F_k]$, decide $Q(f_1, \ldots, f_k) \stackrel{?}{=} 0$.

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where $\mathscr{F} \subseteq \mathbb{B}$ is a differential field with a zero-test.

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Theorem(van der Hoeven 2006)

The roots of any differential polynomial can always be embedded into an algebraic differential extension field generated by exponentials, logarithms, and solutions of several quasi-linear equations.

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where $P_1(f_1) = 0$, $f_1 \prec 1$ is quasi-linear.

Existence

- P: a general differential polynomial in $\mathbb{B}[\delta_1^{\mathbb{N}} F] \setminus \mathbb{B}$
- f: an arbitrary transseries in $\mathbb{B}$ s.t. $f \prec 1$
- $L_{P,f} := (P_{+f})_{[1]}$
- $\tilde{\mathbb{B}}_s := \mathbb{C}[[(\log \circ \overset{s\times}{\cdots} \circ \log \mathfrak{b}_1)^{-\mathbb{R}}; \ldots; (\log \mathfrak{b}_1)^{-\mathbb{R}}; \mathfrak{b}_1^{-\mathbb{R}}; \ldots; \mathfrak{b}_n^{-\mathbb{R}}]]$

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Proposition

Let $P \in \mathbb{B}[\delta_1^{\mathbb{N}} F] \setminus \mathbb{B}$, $f \in \mathbb{B}$ and $\sigma > 0$. Assume that $L_{P,f} \neq 0$ and $f \prec 1$. If $v_n(P(f)) > v_n(L_{P,f}) + \sigma$, then P has a root $\tilde{f}$ in $\tilde{\mathbb{B}}_s$ with $v_n(\tilde{f} - f) > \sigma$, for some $s \in \mathbb{N}$.

Note: We always have $L_{P,f} \neq 0$ whenever $S_P(f) \neq 0$.

Root separation bounds

- P: a polynomial in $\mathbb{B}[\delta_1^{\mathbb{N}} F] \setminus \mathbb{B}$ s.t. $P(f) = 0, f \prec 1$ is quasi-linear
- f: a root of P in $\mathbb{B}$ satisfying $f \prec 1$

Root separation for P at f

Smallest real number $\sigma_{P,f}$ such that

- $\sigma_{P,f} \geqslant v_n(f)$;
- For any $s \in \mathbb{N}$ and $\tilde{f} \in \tilde{\mathbb{B}}_s$, $P(\tilde{f}) = 0 \ \wedge \ v_n(f - \tilde{f}) > \sigma_{P,f} \implies f = \tilde{f}$.

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Proposition

Given a quasi-linear equation $P(f) = 0, f \prec 1$ with a solution f, we have

$$\sigma_{P,f} \leqslant \max(v_n(f), Z_{P,f}).$$

The new algorithm

Let $P(f)=0, f \prec 1$ be a quasi-linear equation with a solution f.

Algorithm ZeroTest$(Q_1,\ldots,Q_s)$
INPUT: $Q_1,\ldots,Q_s \in \mathscr{F}\{F\}\setminus\{0\}$, ordered by non-decreasing Ritt rank
OUTPUT: **true** if $Q_1(f)=\cdots=Q_s(f)=0$ and **false** otherwise

1. If $Q := Q_1 \in \mathscr{F}$ then return **false**
2. If **ZeroTest**(I_Q) then return **ZeroTest**$(I_Q, Q_1,\ldots,Q_s)$
3. If **ZeroTest**(S_Q) then return **ZeroTest**$(S_Q, Q_1,\ldots,Q_s)$
4. If $\exists J \in \{Q_2,\ldots,Q_s,P\}, J \operatorname{rem} Q \neq 0$ then return **ZeroTest**$(J \operatorname{rem} Q, Q_1,\ldots,Q_s)$
5. Let $\sigma := \max(v_n(f), Z_{P,f}, v_n(I_Q(f)), v_n(S_Q(f)))$
6. Return the result of the test $v_n(Q(f)) > \sigma + v_n(L_{Q,f})$

Correctness

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$$I_Q^j S_Q^k P \;=\; U_0 Q + \cdots + U_r \delta_1^r Q$$

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$$I_Q^j S_Q^k P \;=\; U_0 Q + \cdots + U_r \delta_1^r Q \qquad\qquad \exists\, \tilde{f} \in \tilde{\mathbb{B}}_s,\; v_n(\tilde{f} - f) > \sigma \;\wedge\; Q(\tilde{f}) = 0$$

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$$\exists \tilde{f} \in \mathbb{B}_s,\ v_n(\tilde{f} - f) > \sigma \ \wedge \ Q(\tilde{f}) = 0$$
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$$I_Q^j S_Q^k P = U_0 Q + \cdots + U_r \delta_1^r Q$$

$$P(\tilde{f}) = 0$$

$$\exists \tilde{f} \in \tilde{\mathbb{B}}_s,\ v_n(\tilde{f}-f) > \sigma \ \wedge\ \boxed{Q(\tilde{f})=0}$$

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$$
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I_Q^j S_Q^k P &= U_0 Q + \cdots + U_r \delta_1^r Q \\
P(\tilde{f}) &= 0 \\
\tilde{f} &= f
\end{aligned}
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$$v_n(S_Q(\tilde{f})) = v_n(S_Q(f)) < \sigma$$

$$\begin{aligned} P(\tilde{f}) &= 0 \\ \tilde{f} &= f \\ Q(f) &= 0 \end{aligned}$$

Algorithm ZeroTest$(Q_1, \ldots, Q_s)$

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Thank you !

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