Global Curvature of Difference Operators

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July 14, 2026

Linear Difference Operators

A sequence $u \in \mathbb{Q}^{\mathbb{N}}$ is **holonomic** if it satisfies a linear difference equation with *rational function* coefficients. Consider the recurrence:

$$(n+1)\cdot u(n+2) - (2n+3)\cdot u(n+1) + (n+2)(n^2+2n+2)\cdot u(n) = 0$$

We write this equation as $L(u)(n) = 0$ where

$$L = (n+1)\tau^2 - (2n+3)\tau + (n+2)(n^2+2n+2)$$

with τ being the shift operator given by $\tau(u)(n) = u(n+1)$.

L is an element of $R := \mathbb{Q}(n)[\tau]$.

Motivation

The question regarding existence of an algorithm determining whether an operator L has integer solutions $u \in \mathbb{Z}^{\mathbb{N}}$ remains open.

Dwork's Conjecture and Christol's Conjecture are related problems. The first is known to be false and the second has candidate counterexamples.

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Despite this, these conjectures hold for surprisingly many OEIS entries.

Our goal is to find other properties that are common in the OEIS so they may help us understand the open question.

Integer Solutions of Linear Difference Operators

If $u \in \mathbb{Z}^{\mathbb{N}}$ satisfies the difference operator

$$L = (n+1)\tau^2 - (2n+3)\tau + (n+2)(n^2+2n+2)$$

then to compute the next term of the sequence u, one has to divide by the leading coefficient of L:

$$u(n+2) = \frac{2n+3}{n+1} \cdot u(n+1) - \frac{(n+2)(n^2+2n+2)}{n+1} \cdot u(n)$$

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Why does this produces an integer sequence?
(Sequence A105750 in the OEIS)

Integer Solutions of Linear Difference Operators

The following operator A is a left-multiple of L.

$$\begin{aligned} A &= \tau^3 + (n-2)\tau^2 + (n^2+3n+6)\tau + (n+1)(n^2+2n+2) \\ &= \frac{1}{n+2}(\tau + n + 1) \cdot L \end{aligned}$$

This A gives a recurrence of order 3 for u, but since $A \in \mathbb{Z}[n][\tau]$ and A is monic, computing the next term of u only involves dividing by 1, which explains how L can have an integer sequence solution.

The leading coefficient $n+1$ of L is called an **apparent singularity** because it disappears in a left-multiple of L.

Factors of the leading coefficient that cannot be removed in this way are called **true singularities**.

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OEIS entry A005259 is the Apéry sequence. Its minimal difference operator is:

$$L = (n+2)^3\tau^2 - (2n+3)(17n^2+51n+39)\tau + (n+1)^3$$

$(n+2)^3$ is a true singularity: we have to divide by this to compute the next term $u(n+2)$, even if we use another recurrence (a multiple of L).

True Singularities

If $L \in \mathbb{Q}(n)[\tau]$, then let $L \bmod p$ denote its image in $\mathbb{F}_p(n)[\tau]$.

The Apéry operator L has a true singularity $(n+2)^3$, this is an apparant singularity for $L \bmod p$, for every prime p.

This means that in $\mathbb{F}_p(n)[\tau]$, one can find a multiple of our operator $L \bmod p$, where the singularity disappears.

We will study how to prove this and similar properties.

Plan of Attack

The best tool to study an operator $L \in \mathbb{F}_p(n)[\tau]$ is the p-curvature polynomial. It is a multiple of L where every apparant singularity disappears [3, Zhou, van Hoeij - 2022].

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Q: If L is the minimal difference operator of an integer sequence, then for all but finitely many primes, must its p-curvature have no leading singularity?

A: A survey of the OEIS shows this true for more than 99.9% of holonomic sequences.

This motivates us to study the p-curvature further.

The p-curvature polynomial

Let $L \in \mathcal{R} = \mathbb{F}_p(n)[\tau]$, and let $M := \mathcal{R}/\mathcal{R}L$.

Definition 1 (The Linear Map $\tau^p : M \to M$)

$\tau^p : M \to M$ is an $\mathbb{F}_p(n)$-linear map. Its characteristic polynomial is the ***p*-curvature polynomial** of L, denoted $\chi(L)$.

Lemma 2

- $\chi(L) \in \mathbb{F}_p(c)[T]$ *where* $c = n^p - n$*.*
- *For any element* $L \in \mathbb{F}_p(n)^*$*, we have* $\chi(L) = 1$*.*
- *For* $L_1, L_2 \in \mathcal{R}$*, we have* $\chi(L_1 \cdot L_2) = \chi(L_1) \cdot \chi(L_2)$*.*
- L *is a right-hand factor of* $\chi(L)\Big|_{T=\tau^p}$*.*

p-Curvature of L

The p-Curvature for the minimal recurrence of the Apéry sequence is computed in the table below for various primes p.

$$L = (n+2)^3\tau^2 - (2n+3)(17n^2+51n+39)\tau + (n+1)^3$$

p	$\chi(L)$	p	$\chi(L)$	p	$\chi(L)$
2	T^2+1	13	T^2+5T+1	31	$T^2+28T+1$
3	T^2+2T+1	17	T^2+1	37	T^2+3T+1
5	T^2+T+1	19	T^2+4T+1	41	T^2+7T+1
7	T^2+T+1	23	$T^2+12T+1$	43	T^2+9T+1
11	$T^2+10T+1$	29	$T^2+24T+1$	47	$T^2+13T+1$

It appears as though $\chi(L) \in \mathbb{F}_p[T]$ rather than $\mathbb{F}_p(c)[T]$, but why is this?

Ultimately, $\chi(L) \equiv T^2 - 34T + 1 \pmod{p}$ for every prime p.

Definition 3 (Global Curvature)

Let $L \in \mathbb{Q}(n)[\tau]$. We say that L has global curvature if there exists a polynomial $\chi_G(L) \in \mathbb{Q}(c)[T]$ so that for all but finitely many primes, $\chi(L \bmod p) = \chi_G(L) \bmod p$.

We have a heuristic which does a rational reconstruction on the coefficients of the p-curvatures to detect if L has global curvature, and if so, to find it. The more difficult task is to prove that a tentative global curvature is correct.

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Q: How to prove $\chi_G(L) = T^2 - 34T + 1$?

First lets consider a special case.

Nilpotent Linear Differential Operators

Definition 4

Let $D = \mathbb{Q}(x)[\partial]$ be the ring of differential operators with $\partial = \frac{d}{dx}$.

If $L \in D$ then let $L \bmod p \in \mathcal{D} = \mathbb{F}_p(x)[\partial]$ be its image mod p.

$L \bmod p$ is called **nilpotent** if ∂^p acts as a nilpotent linear map on the corresponding module.

L is called **globally nilpotent** if $L \bmod p$ is nilpotent for all but finitely many p.

Theorem on Nilpotent Differential p-Curvatures

Let k be any field. There is a well known isomorphism
$\varphi : k[x, x^{-1}, \partial] \to k[\tau, \tau^{-1}, n]$ where $\varphi(x) = \tau$ and $\varphi(\partial) = (n-1)\tau^{-1}$.

Theorem 5 (van Hoeij, Lawson)

Suppose $L \in \mathbb{Q}[x][\partial]$ is globally nilpotent and that $M := \varphi(L)$ is its corresponding recurrence operator. Then M has global curvature, and moreover, the global curvature is in $\mathbb{Q}[T]$ (rather than $\mathbb{Q}(c)[T]$).

Lemma 6

Let $L = \mathbb{F}_p[n, \tau]$. The coefficients of L of highest degree in n provide the coefficients of the p-curvature of highest degree in c.

The above lemma fully provides the (global) curvature whenever it does not contain c.

Example where the (global) curvature does contain c

Consider $L = \tau^2 - (n+2)\tau - 2 \in \mathbb{Q}(n)[\tau]$.
A broader version of Lemma 6 from the paper gives that, for any prime p, the p-curvature of L will be of the form $T^2 - (c + ?)T - 2$.

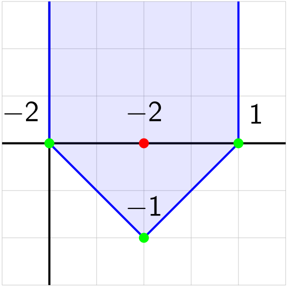
Idea: View $c + ? \in \mathbb{F}_p[c] \subset \mathbb{F}_p((\frac{1}{c}))$ as a Laurent series in $\frac{1}{c}$, then we need the top two terms.

To reduce the problem from order 2 to order 1 we want to factor L.

However, L is irreducible in $\mathbb{Q}(n)[\tau]$.

So we will use a field extension $\mathbb{Q}(n) \subset \mathbb{Q}((t))$ where $t = \frac{1}{n}$.

Newton Polygons and Hensel Lifting



$L = L_a \cdot L_b \in \mathbb{Q}((t))[\tau]$.
The Hensel lifting procedure computes coefficients of L_a and L_b by solving linear equations obtained from the condition:
$L \equiv L_a L_b \bmod t^k$ for ever increasing k.

The Factors L_a and L_b

$$L = \tau^2 - (n+2)\tau - 2 = L_a \cdot L_b$$

$$L_a = \tau + 2t - 2t^2 - 2t^3 + (2\cdot 3)t^4 + (2\cdot 3)t^5 - (2\cdot 3^2)t^6 - (2\cdot 17)t^7 + \dots$$

$$L_b = \tau - n - 1 - 2t + 2^2t^3 + 2^2t^4 - (2^2\cdot 3)t^5 - (2^2\cdot 11)t^6 - (2^2\cdot 7)t^7 + \dots$$

(recall that $t = 1/n$)

Note that $L_a, L_b \in \mathbb{Q}((t))[\tau]$ are actually in $\mathbb{Z}((t))[\tau]$. So their images mod p are well defined, and can thus be used to compute the p-curvature of $L = L_aL_b$.

This way, computing the p-curvature of L for every prime p has now been reduced from order two to order one.

The p-curvature of an operator of order 1

If $L = \tau - f$ then $\tau^p \equiv N_\tau(f) \mod L$, where

$$N_\tau(f) := \prod_{i \in \mathbb{F}_p} \tau^i(f).$$

So the p-curvature of L will be given by:

$$\chi(L \bmod p) = T - N_\tau(f)$$

.

The image of N_τ

Definition 7

$\langle \tau \rangle \cong C_p$ is the Galois group of the extension $\mathbb{F}_p(n)$ over $\mathbb{F}_p(c)$. It is also the Galois group of $\mathbb{F}_p((t))$ over $\mathbb{F}_p((1/c))$. The map N_τ is a field norm:

$$N_\tau : \mathbb{F}_p((t)) \to \mathbb{F}_p((1/c))$$

$$N_\tau(f) = \prod_{i \in \mathbb{F}_p} \tau^i(f)$$

Proving Global Curvature for L

Theorem 8 (van Hoeij, Lawson)

If $g \in \{1 + t \cdot f \mid f \in \mathbb{F}_p[[t]]^*\}$, then $N_\tau(g) = 1 + 0 \cdot \frac{1}{c} + \dots$

$L_a = \tau + t(2 - 2t + \dots)$ and $L_b = \tau - n(1 - t + \dots)$

$$\begin{aligned}\chi_G(L_a) &= T - N_\tau(2t(1 - t + \dots)) \\ &= T - 2\frac{1}{c}\left(1 + 0 \cdot \frac{1}{c} + \dots\right) \\ &= T - 2\frac{1}{c} + 0 \cdot \frac{1}{c^2} + \cdots\end{aligned}$$

$$\begin{aligned}\chi_G(L_b) &= T - N_\tau(n(1 - t + \dots)) \\ &= T - c\left(1 + 0 \cdot \frac{1}{c} + \dots\right) \\ &= T - c + 0 \cdot c^0 + \cdots\end{aligned}$$

$$\chi(L_a) \cdot \chi(L_b) = T^2 - cT - 2 + O\left(\frac{1}{c}\right)$$

Result

$$\chi(L_a) \cdot \chi(L_b) = T^2 - cT - 2 + O\left(\frac{1}{c}\right)$$

Because $L = \tau^2 - (n+2)\tau - 2 = L_a \cdot L_b \in \mathbb{F}_p[n, \tau]$, we have that $\chi_G(L_a) \cdot \chi_G(L_b) = \chi_G(L) \in \mathbb{F}_p[c, T]$ which has no $\frac{1}{c}$ terms.

This proves that $\chi_G(L) = T^2 - cT - 2$.

Proving Global Curvature

By taking the primitive part, we can reduce $\chi(L)$ to $\mathbb{F}_p[c][T]$.
Let d be its degree with respect to c.

- Theorem 5 covers the globally nilpotent case where $d = 0$.
- If $d \leq 1$, then we can try to reduce to order 1 as in the last example.
- If $d \geq 2$, we will need another approach.

Solution for $\deg_c(\mathcal{L}) \geq 2$

OEIS A307376 has minimal difference operator

$$L = 2(9n+4)(n+2)(n+1)\tau^2 - (n+1)(243n^3 + 837n^2 + 900n + 280)\tau$$
$$+\ 8(9\mathrm{n}+13)(2\mathrm{n}+1).$$

The heuristic gives the following global curvature, where we took the primitive part (multiplied by c) to prevent a denominator:

$$cT^2 - \left(\frac{27}{2}c^2 + 2\right)T + 8.$$

We use NormalForm (available in Maple 2026) to reduce L to an equivalent operator which has the differential operator:

$$(54x - 81)\partial^2 - (16x + 30)\partial + 2(4x^3 + 28x^2 - 19x + 6)/x^3$$

which we can solve in terms of Bessel functions with Maple's hypergeometricsols. Bessel Operator has global curvature.

Global Curvature OEIS Survey

Of more than 390,000 entries in the OEIS:

- We identified 55,979 entries as likely satisfying a linear recurrence.
- Of the holonomic sequences, 52,007 entries (92.90%) have at most geometric growth, this case is addressed by our theorem.
- We used a heuristic for the remaining entries. 3,838 (or 96.66%) of remaining entries likely have global curvature. Proving this can be difficult.
- This leaves 134 entries that, according to the heuristic, likely do not have global curvature. This comprises only 0.24% of the holonomic entries. For 99.76% of holonomic OEIS entries, there is a global curvature.

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An Example Without Global Curvature

OEIS Entry A284230 is the sequence whose n^{th} term is the, “number of self-avoiding planar walks starting at (0,0), ending at $(n, 0)$, remaining in the first quadrant and using steps (0,1), (1,0), (1,1), (-1,1), and (1,-1) with the restriction that (0,1) is never used below the diagonal and (1,0) is never used above the diagonal”.

The OEIS lists the following (conjectured) recurrence:

$$\tau^4 - \tau^3 - (n^2 + 9n + 17)\tau^2 - (n + 2)\tau + (n + 2)(n + 1)$$

The corresponding differential equation is solvable in terms of HeunC functions, but not in terms of hypergeometric functions.

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Thank you!