Relating the computational and logical difficulty of solving ODEs: from polynomials to discontinuous right-hand sides

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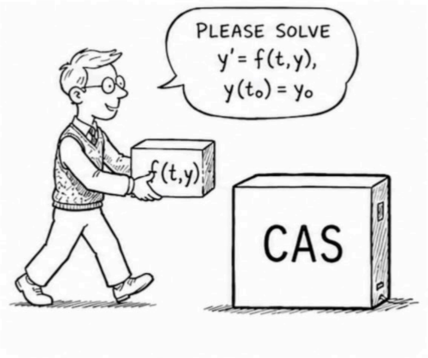
Bug or theorem



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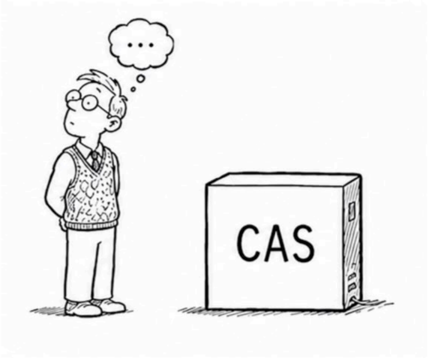

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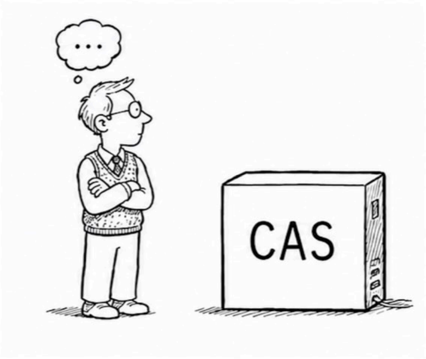

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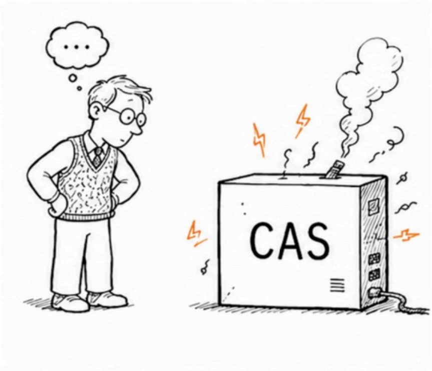

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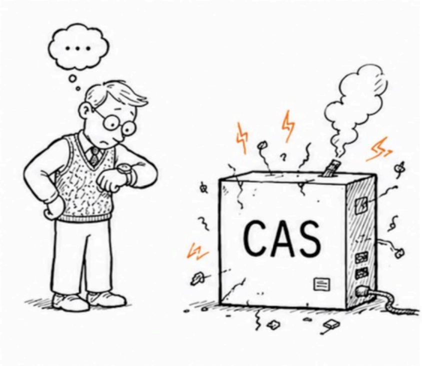

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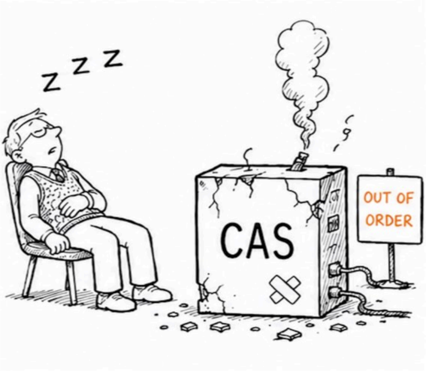

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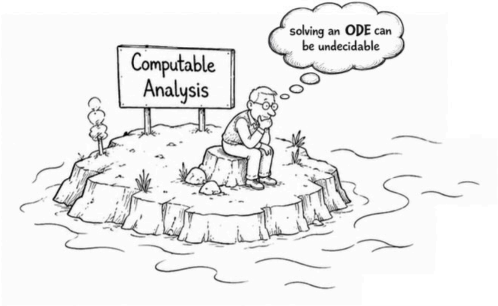

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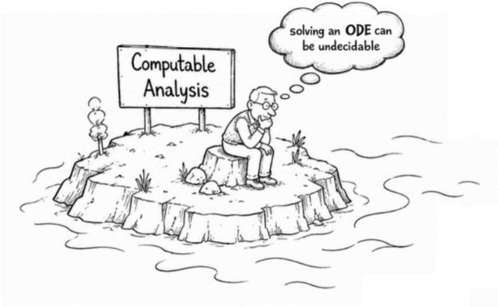

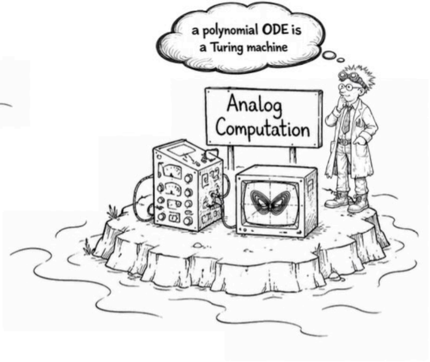

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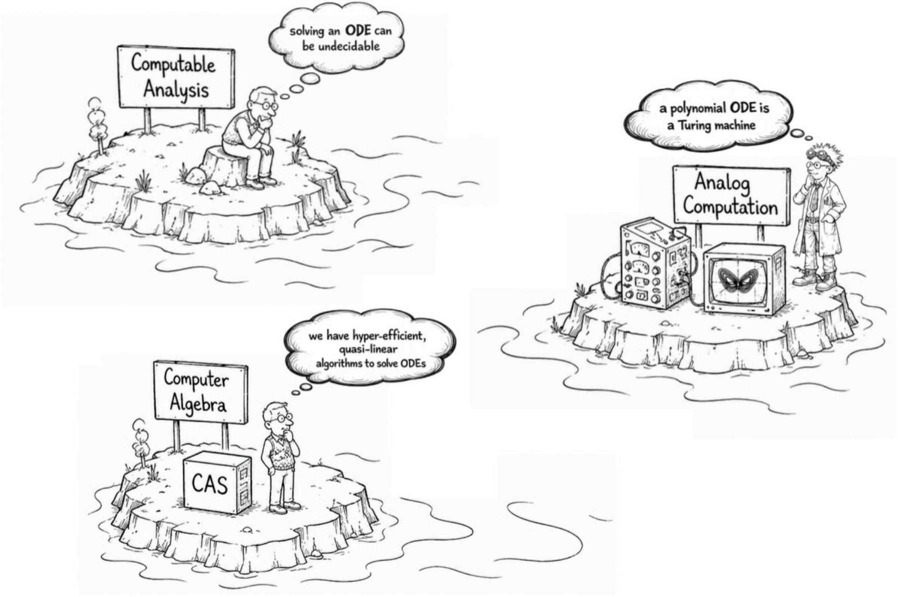

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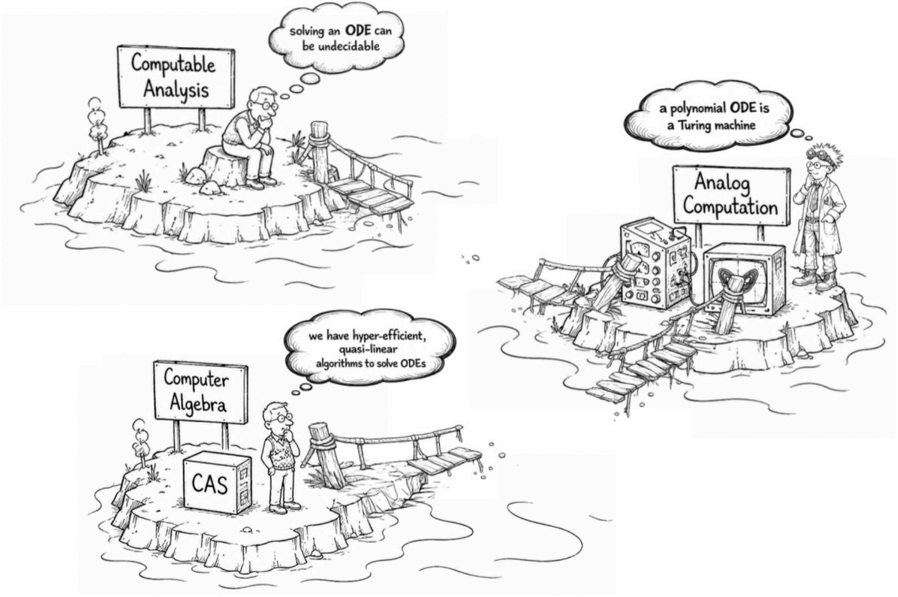

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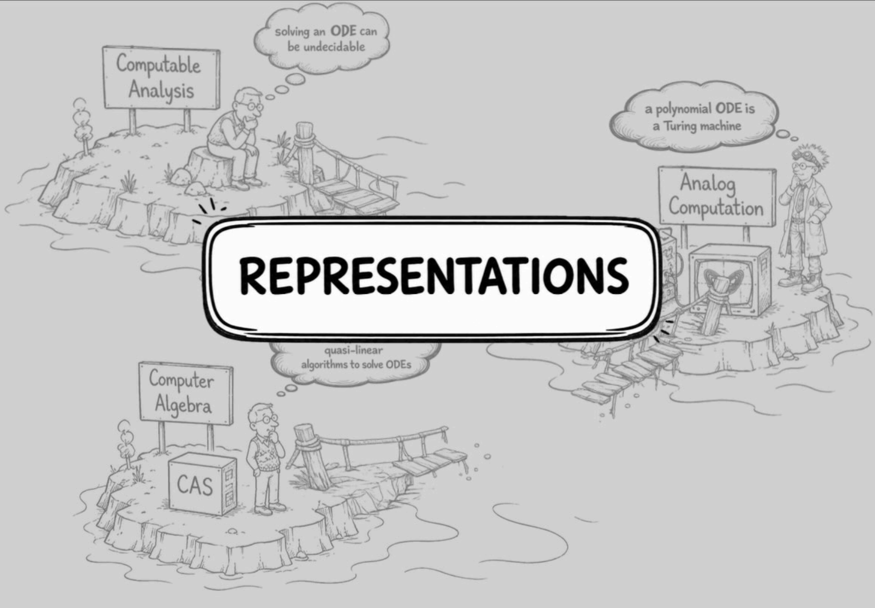

Bug or feature

Plan

We propose a different angle to overcome this.

- Why are some ODEs harder than others?
- Where exactly does this difficulty appear?
- A representation-free paradigm
- A Research Programme

The ODE problem

ODE

Let $\mathcal{I} \times \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^m$ be a compact domain.
Let $f : \mathcal{I} \times \mathcal{D} \to \mathbb{R}^d$ be a __________ map.
An **initial value problem** (IVP) is a

$$\begin{cases} y'(t) = f(t, y), & t \in \mathcal{I} \\ y(t_0) = y_0 & y_0 \in \mathcal{D}, t_0 \in \mathcal{I}. \end{cases}$$

Solution

We call **solution** any map $y : \mathcal{I}' \to \mathbb{R}^d$ that satisfies the IVP.

The ODE problem

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Let $\mathcal{I} \times \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^m$ be a compact domain.
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Let $f : \mathcal{I} \times \mathcal{D} \to \mathbb{R}^d$ be a _Lipschitz_ map.
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Let $f : \mathcal{I} \times \mathcal{D} \to \mathbb{R}^d$ be a "Osgood" map.
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Question

When does a solution exist? and why?

Lipschitz vs Osgood

Theorem (Cauchy-Lipschitz)

If f is Lipschitz in its both variables, the IVP has a unique local solution

Theorem (Cauchy-Osgood)

If f is continuous and satisfies, for a modulus ω, with $\omega(0) = 0$,

$$\|f(t,y) - f(t,z)\| \leq \omega(\|y - z\|), \quad \int_0^{\epsilon} \frac{dr}{\omega(r)} = +\infty,$$

the IVP has a unique local solution.

- Same structure $(\omega(r) = Lr$ recovers Lipschitz$)$
- The same proof structure... almost

“Same” proof

Strategy

Given f and its domain, we:

- Find M such that $\|f(t, y)\| \leq M$ in its domain.
- Use M to define the time interval where the solution will live.
- Build explicit piecewise-linear (Euler-type) approximants y_k.
- Show that $(y_k)_k$ is uniformly Cauchy via an explicit estimate.
- Pass to the limit; uniqueness from a Grönwall-type inequality.

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The Weak König Lemma (a statement about trees)

Compactness in disguise

The Weak König Lemma

Every infinite binary tree has an infinite path

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Compactness disguised as a tree lemma

Heine-Borel: every open cover of $[0,1]$ has a finite subcover.

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Non-constructivity at its finest! and at its weakest
This is optimal: Heine-Borel $\iff$ WKL

The one sentence where constructivity dies

The M

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The rest

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For free

Cauchy-Osgood

“There exists M...”, this requires:

- A compactness argument (à la Heine-Borel)
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This is not an accident... it is a programme!

Mathematics

Reverse Mathematics

Reverse Mathematics

A representation-free paradigm

What is it?

A programme that asks a specific question about every theorem:

“Exactly which set-axioms are necessary (and sufficient) to prove this theorem?”

- The logical strength of a theorem does not depend on the objects’ representation
- Logical strength is a property of the mathematical content, not the computational packaging

The Big Five

The systems

1. **$\mathbf{RCA}_0$:** the base system of computable mathematics
2. **$\mathbf{WKL}_0$:** adds the Weak König Lemma (Compactness)
3. **$\mathbf{ACA}_0$:** the Turing jump of any set exists
4. **$\mathbf{ATR}_0$:** Arithmetic Transfinite Recursion
5. Π^1_1-**$\mathbf{CA}_0$:** Comprehension for Π^1_1 formulas

$$\mathrm{RCA}_0 \subsetneq \mathrm{WKL}_0 \subsetneq \mathrm{ACA}_0 \subsetneq \mathrm{ATR}_0 \subsetneq \Pi^1_1\text{-}\mathrm{CA}_0$$

Populating the Big five

RCA_0: Cauchy-Lipschitz [Simpson]

If $f : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^d$ is Lipschitz in both variables, the IVP has a unique local solution

WKL_0: Cauchy-Osgood [Bournez & Yours truly]

If $f : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^d$ is Osgood, the IVP has a unique local solution

ACA_0: Cauchy-Peano maximal [Bournez & Yours truly]

If $f : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^d$ is continuous, the IVP has a maximal solution

ATR_0: Bounded Cauchy-Bournez-Gozzi [Bournez & Yours truly]

If the IVP has a unique solution and $f : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^d$ is α-solvable, then the solution is definable

Π^1_1-CA_0: Cauchy-Bournez-Gozzi [Bournez & Yours truly]

If the IVP has a unique solution and $f : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^d$ is solvable, then the solution is definable

Take-away

- ODEs are hard to solve
- Usual "Complexity measures" depend on descriptions
- Mathematics does not
  - it captures the logical strength of the mathematical content
  - it is immune to bad descriptions
- The "ultimate model" for complexity-theoretical applications within the computable world remains to be found
  - Bounded Reverse Maths


That's all Folks!
Thank you