Asymptotic properties of random monomial ideals

Fatemeh Mohammadi Sonja Petrović Eduardo Sáenz-de-Cabezón
KU Leuven Illinois Institute of Technology Universidad de La Rioja

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Motivation

- Generalization of results in graph and network theory
- Statistical counterpart of usual ideal-by-ideal study
- Applications to sensitivity analysis, system reliability, etc.

Related work

- Random network models

  Paul Erdős and Alfréd Rényi (1959). "On Random Graphs. I". In: *Publicationes Mathematicae* 6.3-4, pp. 290–297.

- Random monomial ideals

  Jesús A. De Loera et al. (2019b). "Random monomial ideals". In: *Journal of Algebra* 519, pp. 440–473.

- Lcm lattice in applications

  Fatemeh Mohammadi, Eduardo Sáenz-de-Cabezón, and Henry Wynn (2025). "Redundancy analysis using lcm-filtrations: networks, system signature and sensitivity evaluation". In: *Proceedings of the 2025 International Symposium on Symbolic and Algebraic Computation*, pp. 293–301.

Main object of study: LCM-lattice density

Let $M = \{\mathbf{x}^{\mu_1}, \ldots, \mathbf{x}^{\mu_r}\}$ be a finite set of monomials in $R = \mathbf{k}[x_1, \ldots, x_n]$. For each variable x_i, let α_i be the maximum exponent of x_i appearing among the monomials in M. The least common multiple of the elements of M is then $\operatorname{lcm}(M) = \prod_{i=1}^{n} x_i^{\alpha_i}$.

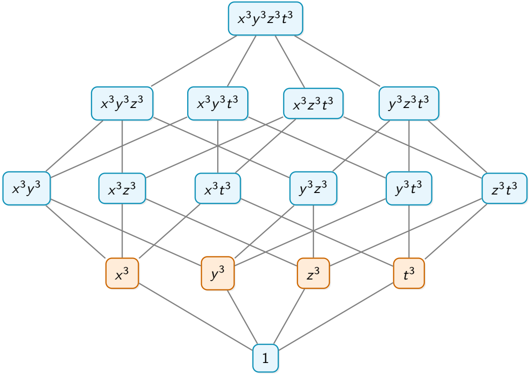
Definition

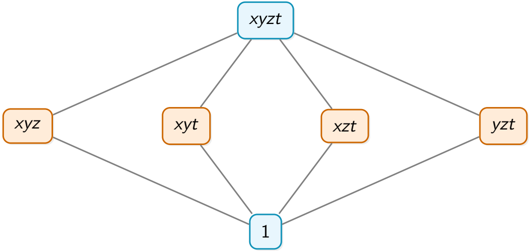
Let $I \subseteq \mathbf{k}[\mathbf{x}]$ be a monomial ideal with minimal monomial generating set $G(I) = \{m_1, \ldots, m_r\}$. The lcm-*lattice* of I is the set

$$L_I = \left\{\operatorname{lcm}(\{m_i : i \in \sigma\}) \;\middle|\; \sigma \subseteq \{1, \ldots, r\}\right\},$$

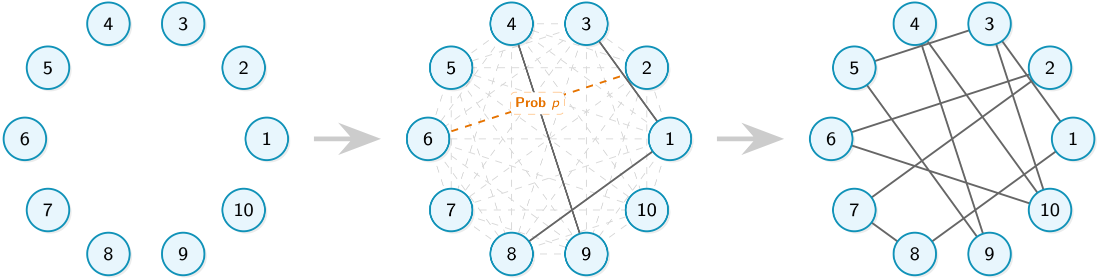
partially ordered by divisibility. This poset is a finite atomic lattice, with atoms corresponding to the minimal generators of I.

Main object of study: LCM-lattice density





The model: Extension of Erdős-Renyi model



Erdős-Renyi model: asymptotic properties

Degree distribution
For large n with average degree $\langle k \rangle$ fixed, the degree distribution converges to a Poisson distribution [Newman 2018]:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}. \tag{1}$$

Thus, most vertices have degree close to the average, while vertices of very high degree are exponentially rare [Barabási 2016].

Erdős-Renyi model: asymptotic properties

Clustering coefficient

The local clustering coefficient is defined as $C = p = \langle k \rangle / n$. Consequently, for fixed $\langle k \rangle$ and $n \to \infty$, we have $C \to 0$. This reflects the locally sparse nature of Erdős–Rényi graphs and their lack of the high clustering typically observed in real-world networks [Newman 2018].

Erdős-Renyi model: asymptotic properties

Small-world property
Despite low clustering, the average path length L grows only logarithmically with the number of vertices:

$$L \sim \frac{\ln n}{\ln \langle k \rangle}. \tag{2}$$

This logarithmic scaling is known as the *small-world* effect [Barabási 2016].

Erdős-Renyi model: Phase transition

A particularly important property of the Erdős–Rényi model is the abrupt change in its global connectivity as the edge probability p increases.
This is a continuous (second-order) phase transition, commonly referred to as the *percolation transition* [Erdős and Rényi 1960].
It is reflected in the asymptotic behavior of the size N_G of the largest connected component relative to the total number of vertices n [Erdős and Rényi 1960; Bollobás 2001].

Erdős-Renyi model: Phase transition

- **Subcritical Phase ($\langle k \rangle < 1$):** The graph consists of many small, disconnected components. The largest connected component has logarithmic size, $N_G = O(\log n)$, and the graph is asymptotically a forest of small trees.
- **Critical Point ($\langle k \rangle = 1$):** At the threshold of the phase transition, the largest connected component grows rapidly and has size $N_G \sim n^{2/3}$.
- **Supercritical Phase ($\langle k \rangle > 1$):** A unique giant connected component (GCC) emerges, occupying a positive fraction of the vertices, $N_G \sim S \cdot n$ with $S > 0$. All remaining components are microscopic, of size $O(\log n)$.

Erdős-Renyi model: Phase transition

Remark

It is important to distinguish the existence of a giant component from full connectivity of the graph. The network becomes fully connected (no isolated nodes remaining) only when:

$$p > \frac{\ln n}{n}. \tag{3}$$

This threshold is much higher than that of the percolation transition ($p > 1/n$) [Erdős and Rényi 1959; Bollobás 2001].

Structural sets of monomials

Definition

Let $\mathfrak{U}$ be a set of monomials in R. We say that $\mathfrak{U}$ is a *structural set of monomials* if every nonempty subset $\mathcal{S} \subseteq \mathfrak{U}$ is the minimal generating set of a monomial ideal in R, i.e. for every pair $\mathbf{x}^\mu \neq \mathbf{x}^\nu$ in $\mathcal{S}$, we have that $\mathbf{x}^\mu \nmid \mathbf{x}^\nu$ and $\mathbf{x}^\nu \nmid \mathbf{x}^\mu$.
For every set $\mathcal{S} \subseteq \mathfrak{U}$ we denote by $I_\mathcal{S}$ the monomial ideal minimally generated by the elements of $\mathcal{S}$.

Density of structural sets of monomials

Definition

Let $\mathfrak{U} \subseteq R$ be a structural set of monomials, and $\mathcal{S} \subseteq \mathfrak{U}$. The *combinatorial density* of $\mathcal{S}$ is given by

$$\mathrm{cd}(\mathcal{S}) = \frac{|\mathcal{S}|}{|\mathfrak{U}|},$$

and the *algebraic density* of $\mathcal{S}$ is given by

$$\mathrm{ad}(\mathcal{S}) = \frac{|L_{I_{\mathcal{S}}}|}{2^{|\mathcal{S}|}},$$

where $L_{I_{\mathcal{S}}}$ is the lcm-lattice of $I_{\mathcal{S}}$. We shall also use the term *poset density* or lcm-*density* for the algebraic density.

Density of structural sets of monomials

Definition

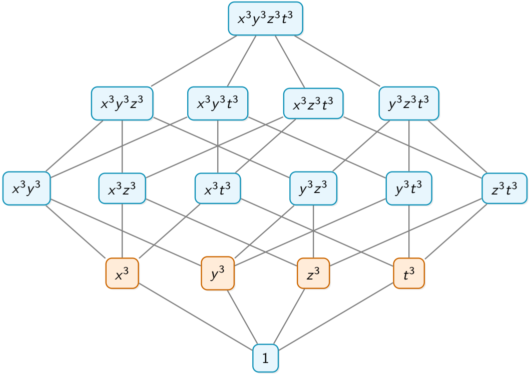
Let $I \subseteq \mathbf{k}[x_1 \ldots, x_n]$ be a monomial ideal. The *Betti density of* I, $\mathrm{bd}(I)$ *is defined by*

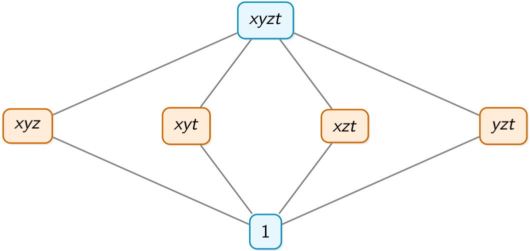
$$\mathrm{bd}(I) = \frac{\sum_{\beta_i(I) \neq 0} \beta_i(I)}{2^{|\mathcal{S}|}},$$

and the length Betti density *of* I *is defined by*

$$\mathrm{lbd}(I) = \frac{|\{j \mid \beta_{i,j}(I) \neq 0\}|}{2^{|\mathcal{S}|}},$$

Density of structural sets of monomials





Density of structural sets of monomials

Definition

A *random model* on a structural monomial set $\mathfrak{U}$ is a stochastic process that randomly selects a subset $\mathcal{S} \subseteq \mathfrak{U}$, and hence a monomial ideal $I_{\mathcal{S}}$.

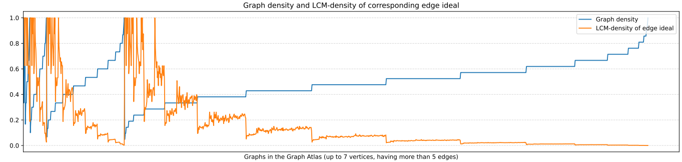
In this way, randomness provides a mechanism for probing typical interaction patterns rather than properties of individual ideals.

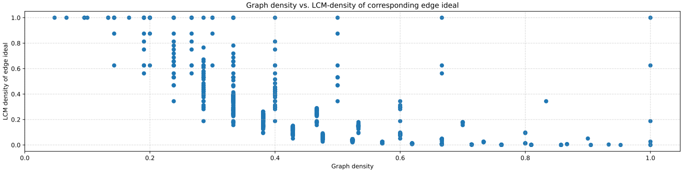
Density of structural sets of monomials

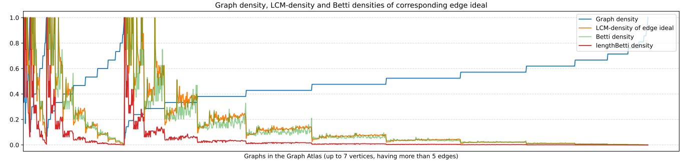
Examples

The structural set of monomials $\mathfrak{U}_{(1,n,2)}$ is in one-to-one correspondence with the set of all simple graphs on n vertices, $\mathfrak{G}_n$.
To each graph $G = (V, E)$ in $\mathfrak{G}_n$ we assign its corresponding monomial ideal $I_G = \langle x_i x_j \mid (i, j) \in E \rangle$.
Conversely, every squarefree monomial ideal in n variables that is equigenerated in degree 2 is the monomial edge ideal of some graph in $\mathfrak{G}_n$.
In this case, the combinatorial density of a set $\mathcal{S} \subseteq \mathfrak{U}_{(1,n,2)}$ coincides with the usual graph density of the corresponding graph $G = (V, \mathcal{S})$, which we denote by $gd(\mathcal{S})$ or $gd(G)$.

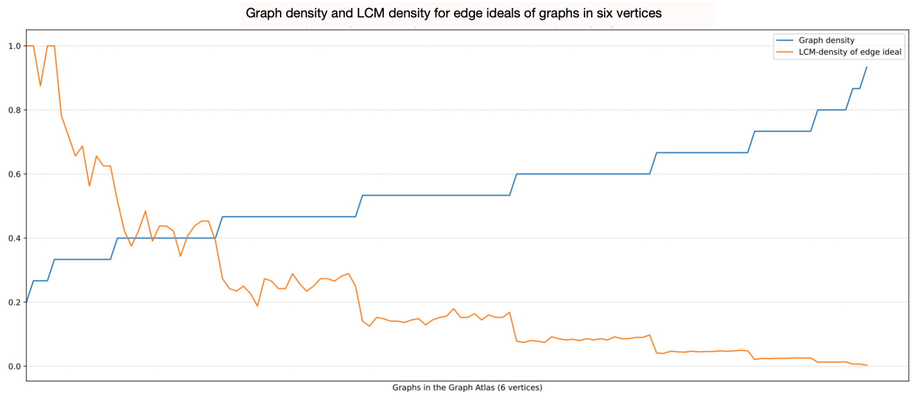
Density of structural sets of monomials



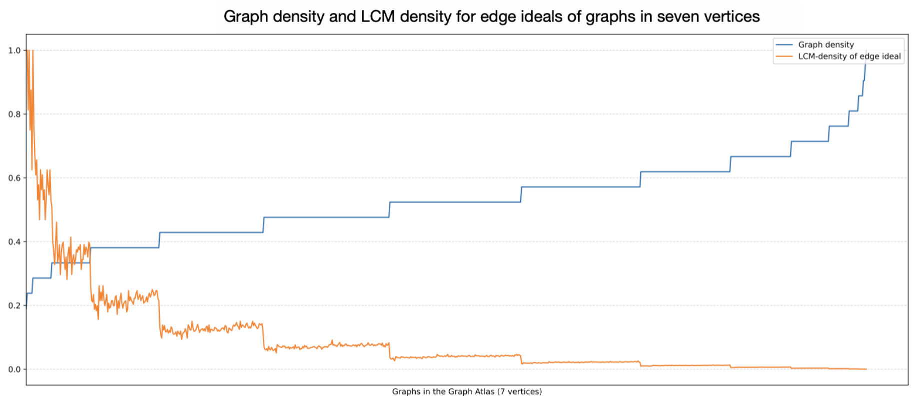




Density of structural sets of monomials



Density of structural sets of monomials



Density of structural sets of monomials

Conjecture

The lcm-density of edge ideals of graphs exhibits a phase transition phenomenon when considering the set of all graphs on n vertices as $n \to \infty$.

The Erdős–Rényi model applies verbatim to $\mathfrak{U}_{(1,n,2)}$, and Theorem 3.10 in [Mohammadi, Sáenz-de-Cabezón, and Wynn 2025] shows that Conjecture 1.1 holds for the family of $ER(n, p)$-model graphs (or, equivalently, the corresponding monomial ideals).

Erdős-Rényi model for $\mathfrak{U}_{(1,n,d)}$, $d > 2$

The Erdős–Rényi model can be directly generalized to $\mathfrak{U}_{(1,n,d)}$ for $d > 2$, and many of its properties admit natural analogues.

For each element of $\mathfrak{U}_{(1,n,d)}$, that is, for each d-subset $\sigma \subseteq \{1, \ldots, n\}$, include σ in $\mathcal{S}$ independently with probability p.

The expected size of $\mathcal{S}$ is $p \cdot \binom{n}{d}$. If we define the density of a collection $\mathcal{S}$ by

$$\mathrm{cd}(\mathcal{S}) = \frac{|\mathcal{S}|}{\binom{n}{d}},$$

then the expected density of $\mathcal{S}$ is p.

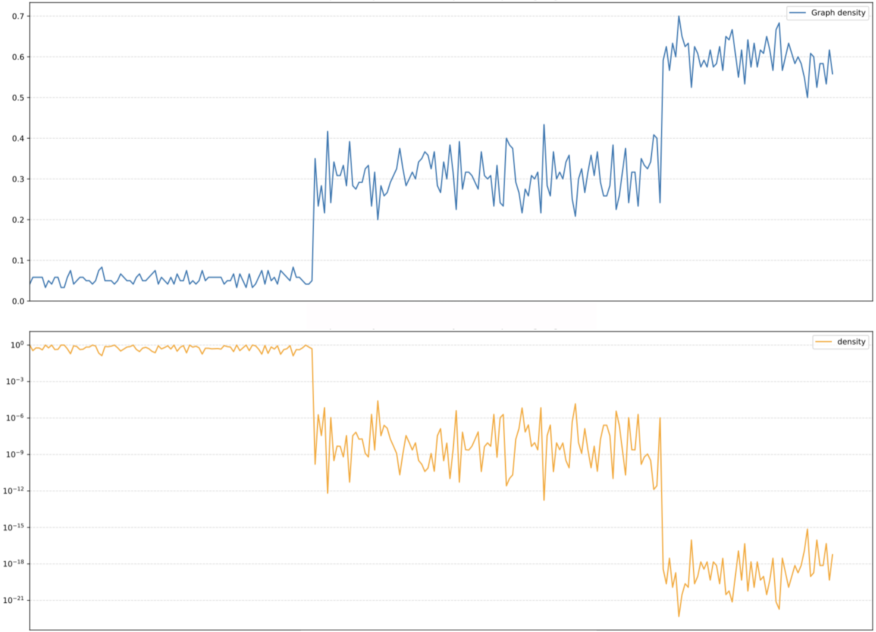
Erdős-Rényi model for $\mathfrak{U}_{(1,n,d)}$, $d > 2$

For any d-subset $\sigma \subseteq \{1, \ldots, n\}$, define the monomial $\mathbf{x}_\sigma = \prod_{i \in \sigma} x_i$, which has degree d. The monomial ideal $I_\mathcal{S} = \langle \mathbf{x}_\sigma \mid \sigma \in \mathcal{S} \rangle$ is a squarefree monomial ideal equigenerated in degree d, and every such ideal arises in this way. The monomial ideals in this set correspond to edge ideals of d-uniform hypergraphs.

Erdős-Rényi model for $\mathfrak{U}_{(1,n,d)}$, $d > 2$

- There is a phase transition for the lcm-density of equigenerated squarefree monomial ideals.
- The logarithm of the lcm-density of equigenerated squarefree monomial ideals exhibits a strong negative correlation with the density of the corresponding collection of generators.

Graph density and LCM-density of corresponding edge ideal



This yields

$$|L_{I_{\mathcal{S}}}| \approx 2^{r\left(1+\frac{\alpha}{\binom{n}{d}}\right)+\beta},$$

showing that the size of the lcm-lattice of equigenerated squarefree monomial ideals of degree d depends only on the number of generators.

General random models and parametric families of ideals

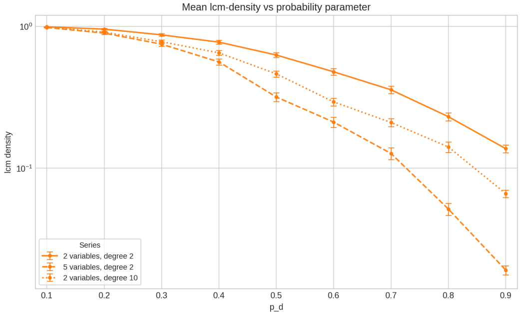
The Erdős–Rényi model for squarefree equigenerated monomial ideals can be seen as a special case of the random monomial ideal model introduced in [De Loera et al. 2019a], which was also inspired by the study of random graphs and simplicial complexes.

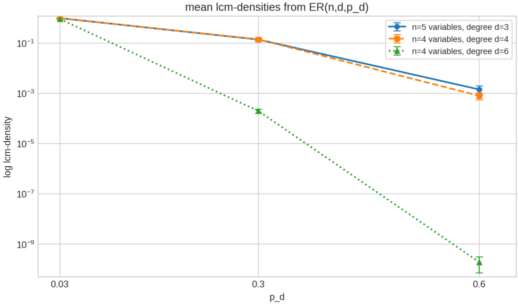
In [De Loera et al. 2019a], the authors define the *graded* model, which allows the probability parameter p to be a vector of length D, so that each degree i of a minimal generator is associated with a probability p_i. The behavior of general random monomial ideals appears to mimic that of graphs and squarefree ideals.

General random models and parametric families of ideals

These results indicate two things:

- The Erdős–Rényi-type random model for monomial ideals captures the lcm-density behavior.
- The sample mean of lcm-density of equigenerated random monomial ideals is a good predictor for the behavior shown in the previous figure, for smaller number of variables n.





Thank you for your attention.