SAGBI and Gröbner Bases Detection

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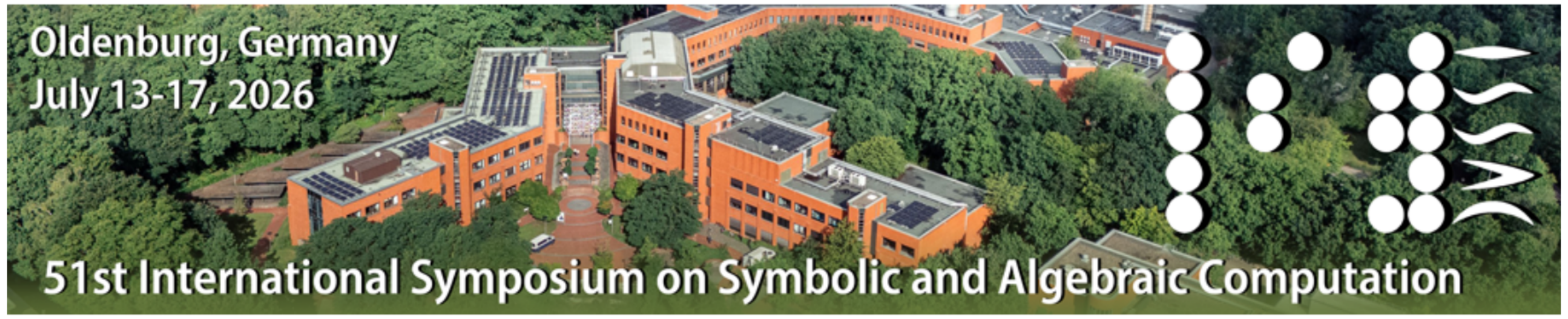
joint work with Viktoriia Borovik and Timothy Duff



Definitions to recall

TERM ORDER / WEIGHT

A weight scores each monomial.
Keep the terms with largest score.

$$f = \sum_{\alpha} c_\alpha x^\alpha, \qquad M = \max\{w \cdot \alpha : c_\alpha \neq 0\}$$
$$\mathrm{in}_w(f) = \sum_{w \cdot \alpha = M} c_\alpha x^\alpha$$

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GRÖBNER BASIS

Let I be an ideal and G a finite subset of I.
G is Gröbner when its leading terms
generate the initial ideal.

$$\mathrm{in}_w(I)=\langle\, \mathrm{in}_w(g) : g\in G \,\rangle$$

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SAGBI BASIS

The w-leading forms generate the w-initial algebra.

$$\mathrm{in}_w(K[F]) = K[\mathrm{in}_w(f_1), \ldots, \mathrm{in}_w(f_s)]$$

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DETECTION QUESTION

For which term orders is F a Gröbner or SAGBI basis?

$$\{\, w :\ F \text{ is GB or SAGBI} \,\} \quad \text{or} \quad \varnothing$$

Why it is hard, and the geometry that helps

THE OBSTACLES

- Infinitely many orders. One fixed-order test is finite.
- A finite SAGBI basis may not exist.

standard warning example

$$K[x+y,\ xy,\ xy^2]$$

THE GEOMETRY

Weights $w \in \mathbb{R}^n$ encode term orders.

F-equivalence means $\mathrm{in}_w(f_i)$ agrees for every $f_i \in F$.

Gritzmann–Sturmfels (1993)
Equivalence classes are normal cones.

$$\mathcal{C}_F = \{\, w :\ \mathrm{in}_w(F) \text{ is fixed} \,\}$$

BOROVIK–DUFF–SHEHU
Finite geometry drives the SAGBI search.

Infinite search → finite normal-cone walk.

A finite sweep over cones

ALGORITHM 1 · GRÖBNER DETECTION

$$\mathrm{New}(G) = \mathrm{New}(g_1 \cdots g_s) = \mathrm{New}(g_1) + \cdots + \mathrm{New}(g_s)$$

$\mathrm{New}(G)$ polytope → C_v normal cone → w representative → Buchberger criterion

GRITZMANN–STURMFELS (1993)

Finite cone sweep on the ideal side.

Same geometric sweep, different algebraic test

COMMON GEOMETRIC STAGE

$$\mathrm{New}(\,\cdot\,) \longrightarrow \{C_v\} \longrightarrow w$$

one representative weight per cone

GRITZMANN–STURMFELS

$$G \longrightarrow \langle G \rangle$$

Buchberger

BOROVIK–DUFF–SHEHU · SAGBI DETECTION

$$F \longrightarrow K[F]$$

Toric relations + subduction

Borovik–Duff–Shehu keep the sweep and apply the SAGBI test at each representative weight.

Toric relations and subduction

For a fixed weight, use the leading-exponent matrix.

$$I_A = \ker(K[y_1, \dots, y_s] \to K[x_1, \dots, x_n])$$
$$y_i \mapsto \mathrm{in}_w(f_i)$$

Each toric relation gives an S-polynomial in the subalgebra.

SAGBI CRITERION

Every toric S-polynomial subduces to 0.

Test any finite generating set of the toric ideal.

SAGBI CRITERION
Toric relations plus subduction give the fixed-order test.

BOROVIK–DUFF–SHEHU
This test sits inside the finite detection sweep.

Julia verifies this property through equivalent exact checks.

Two exact SAGBI verification criteria

INITIAL-IDEAL CRITERION

With the relation ideal

$$I = \ker(y_i \mapsto f_i)$$

$$\mathrm{in}_{A^T w}(I) = I_A$$

A single ideal equality.

HILBERT-SERIES CRITERION

General input: homogenize first.
Compare the Hilbert series.

$$S = K[F] \quad \text{and} \quad K[\mathrm{in}_w(F)]$$

Equality gives an exact yes/no test.

Two exact verification routes for one fixed weight.

Algorithm 2: putting it together

1. $\mathrm{New}(F)$ — polytope $\rightarrow$
2. C_v — normal cone $\rightarrow$
3. w — representative $\rightarrow$
4. SAGBI — criterion $\rightarrow$
5. W_{succ} — output

$$\mathrm{New}(F) \longrightarrow \{C_v\} \longrightarrow \{w\} \longrightarrow W_{\mathrm{succ}}$$

A small example shows the whole algorithm

$$F = \{x^2 + y^2,\ xy,\ y^2\}$$

NEWTON POLYTOPE

$$\mathrm{New}(F) = \mathrm{conv}\{(3,3),(1,5)\}$$

MINKOWSKI-SUM ENDPOINTS

$(2,0)+(1,1)+(0,2)=(3,3),\qquad (0,2)+(1,1)+(0,2)=(1,5)$

Two nonnegative cones survive.

One representative weight per cone.

CANDIDATE CLASSES

$$C = \{\, w_1 > w_2 \geq 0 \,\}$$

SAGBI

$$C' = \{\, w_2 > w_1 \geq 0 \,\}$$

fails

The test is run once per cone.

One representative per cone is enough

GEOMETRY

Normal cones encode equivalence classes.

STABILITY

SAGBI success persists inside one class.

A finite sweep sees every successful class.

Theorem: Algorithm 2 is correct.

Fixed-parameter tractable

For fixed n, homogeneous SAGBI detection has an explicit arithmetic bound.

$$\mathcal{O}\left(\binom{s+s^2d^n}{s}\, m^{2^n-1}\, s^{4n^2+11n+4}\, d^{2n^3+8n^2+5n+1}\right)$$

RESULT

A parameter-controlled detection theorem.

PARAMETERS

n = number of variables

s = number of input generators

d = maximum total degree

m = maximum monomials per generator

Polynomial when n and s are fixed.

SagbiGbDetection

TWO IMPLEMENTATIONS

- Macaulay2 package.
- Oscar-native Julia package.

HEADLINE FUNCTIONS

weightVectorsRealizingGB
weightVectorsRealizingSAGBI

SagbiCriterion
BuchbergerCriterion

extractWeightVectors

Empty output means no term order works.

Demo

Thank You

Thank you. I am happy to discuss questions and computations.