Polyhedra and Spectrahedra on fields with valuations

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Table of contents

Table of contents

For $\ell_1, \dots, \ell_m \in \mathbb{R}[x]_{\leq 1}$ real affine *n*-variate functions:

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For $A_0, A_1, \dots, A_n \in \mathbb{S}^m$ real symmetric matrices

Spectrahedron: $\mathscr{S} = \{x \in \mathbb{R}^n : A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0\}$

"$\succeq 0$" = positive semi-definite = its eigenvalues are ≥ 0.
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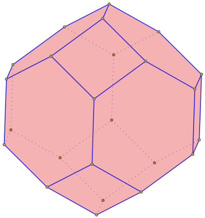
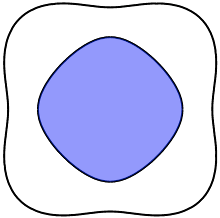
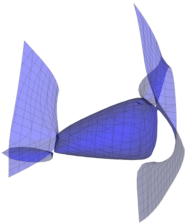
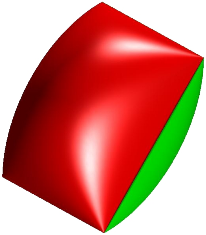
Fact: Polyhedra resp. Spectrahedra are **convex**.

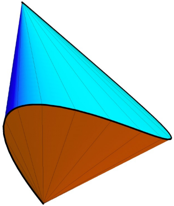
Polyhedra and Spectrahedra over $\mathbb{R}$

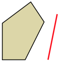


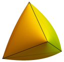


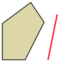


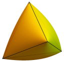




Polyhedra → feasible sets of Linear Programming (LP) 

Spectrahedra → feasible sets of Semidefinite Programming (SDP) 

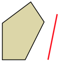
Polyhedra → feasible sets of Linear Programming (LP) 

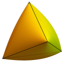
Spectrahedra → feasible sets of Semidefinite Programming (SDP) 

Classical applications from

- real algebraic geometry (positive polynomials vs SOS, 17th Hilbert's problem)
- control theory (Lyapunov stability)
- combinatorial optimization (e.g. SDP relaxation of MAXCUT)
- polynomial optimization (moment-SOS relaxation hierarchy)

Motivations and properties

Polyhedra → feasible sets of Linear Programming (LP) 

Spectrahedra → feasible sets of Semidefinite Programming (SDP) 

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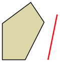
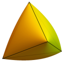
Polyhedra $\to$ feasible sets of Linear Programming (LP) 

Spectrahedra $\to$ feasible sets of Semidefinite Programming (SDP) 

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Several **good properties** besides convex but spectrahedra/SDP are highly more **pathological** than polyhedra/LP:

- spectrahedra are not closed under linear image
- solutions only over algebraic extension: $\{\sqrt{2}\} = \{x \in \mathbb{R} : \left[\begin{smallmatrix} x & 1 \\ 2 & x \end{smallmatrix}\right] \succeq 0$ and $\left[\begin{smallmatrix} 1 & x \\ x & 2 \end{smallmatrix}\right] \succeq 0\}$

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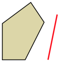
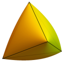
Polyhedra $\to$ feasible sets of Linear Programming (LP) 

Spectrahedra $\to$ feasible sets of Semidefinite Programming (SDP) 

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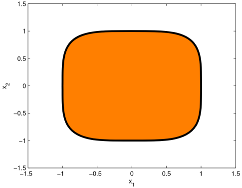

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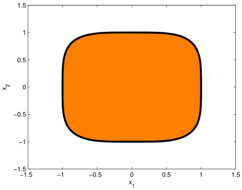

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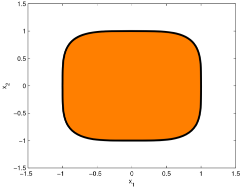

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Not every convex (compact) semialgebraic set is a spectrahedral shadow (Scheiderer, 2018).
But what about spectrahedra over much more exciting fields like $\mathbb{Q}_p$?

Table of contents

Table of contents

Setting

Discrete-valuation fields

Let K be a field with:

- discrete valuation $v_K : K \to \mathbb{R} \cup \{+\infty\}$ making it complete,
- integer ring O_K,
- uniformizer π
- residue field κ.

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Example

K	valuation	R	π	κ
$\mathbb{Q}_p$	p-adic	$\mathbb{Z}_p$	p	$\mathbb{F}_p$
$\mathbb{F}_q((t))$	t-adic	$\mathbb{F}_q[[t]]$	t	$\mathbb{F}_q$
$\mathbb{Q}((t))$	t-adic	$\mathbb{Q}[[t]]$	t	$\mathbb{Q}$

and their finite extensions.

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Topology somehow in-between discrete and continuous mathematics.

Non-archimedean balls

Definition (Ball/disks)

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$$B_{K^n}(\alpha_1, \ldots, \alpha_n; \lambda_1, \ldots, \lambda_n) := \{x \in K^n : \ \forall i, \ v_K(x_i - \alpha_i) \geq \lambda_i\}.$$

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$$B_K(\alpha;\lambda) := \{x \in K : \ v_K(x-\alpha) \geq \lambda\}\,.$$

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Proposition

- *Any point of $D_K(\alpha;\lambda)$ is a center of this disk.*
- *If two disks have a non-trivial intersection, then one contains the other. As such, any set of disks has a forest / tree-like structure for inclusion.*

Annuli

Definition (Annulus)

$$\mathcal{C}_K(\alpha, a, b) = \{x \in K \mid a \leq \mathrm{val}(x - \alpha) \leq b\},$$

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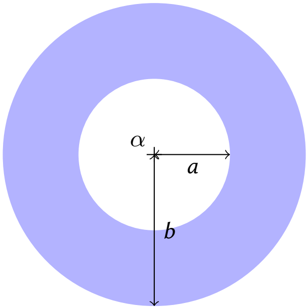

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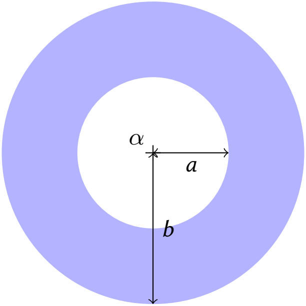

Remark

Central object in p-adic differential equations or in non-archimedean analytic geometry.
Used to define series converging on $\mathcal{C}_K(\alpha, a, b)$.

Table of contents

Table of contents

Non-archimedean polyhedron

Definition

A *polyhedron* is a subset $\mathcal{P} \subset K^n$ of the form

$$\mathcal{P} = \left\{ x \in K^n : \begin{array}{ll} \operatorname{val}(\ell_i(x)) \geq 0 & i = 1, \dots, d \\ \operatorname{val}(m_j(x)) = +\infty & j = 1, \dots, e \end{array} \right\},$$

for some $\ell_1, \dots, \ell_d, m_1, \dots, m_e \in K[x]_1$ (polynomials of degree 1).

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This can be written in matrix form:

$$\mathcal{P} = \{x \in K^n : Ax + v \succeq 0, Bx + w = 0\},$$

for $A \in K^{d \times n}$, $v \in K^d$, $B \in K^{e \times n}$, and $w \in K^e$ such that $Ax + v = (\ell_1, \dots, \ell_d)^T$ and $Bx + w = (m_1, \dots, m_e)^T$.

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Example

$$D(c, \alpha) = \{x \in K^n : val\,(x - c) \geq \alpha\} = \left\{x \in K^n : val\left(\pi^{-\alpha}(x - c)\right) \geq 0\right\}.$$

As such, balls are polyhedron, and similarly for polydisks.

Projecting polyhedron

Theorem (direct image (DI))

Let $f : K^n \to K^m$ be an affine map and $\mathcal{P}$ a polyhedron in K^n. Then $f(\mathcal{P})$ is a polyhedron in K^m.

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Idea of proof.

Use the Smith Normal Form of the linear part f_0 of f.

$$Mat_B(f_0) = P \begin{pmatrix} \pi^{\alpha_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \pi^{\alpha_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \pi^{\alpha_s} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} Q$$

with $P, Q \in GL_n(O_K)$.
Reduce to the case of: translation, isometry, projection. $\square$

Proposition (Characterization of polyhedra)

A non-empty polyhedron is the image of a polydisc under an affine map.

Applications

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Proposition

Polyhedra in K are either the empty set or balls, possibly with infinite radius.

Table of contents

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Corollary

$$O_K^{d\times d} \subset K_+^{d\times d}.$$

(Counter)-Example

Example

Let

$$M = \begin{bmatrix} \frac{3}{5} + 5 & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \in \mathbb{Q}_5^{2\times 2}.$$

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Then

$$\chi_M(T) = T^2 - 5T - 4,$$

Hence

$$M \in (\mathbb{Q}_5)_+^{2\times 2}$$
$$M \notin (\mathbb{Z}_5)^{2\times 2}.$$

Table of contents

What is a spectrahedron?

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A *spectrahedron in* $K^{d\times d}$

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$$\mathcal{S} = \mathcal{L} \cap K_+^{d\times d} = \left\{ X \in K^{d\times d} : \exists\, x \in K^n, X = A_0 + \sum_{i=1}^{n} x_i A_i, X \succeq 0 \right\}$$

where $\mathcal{L} = A_0 + \operatorname{span}(A_1, \ldots, A_n) \subset K^{d\times d}$ is an affine space, for some $A_0, A_1, \ldots, A_n \in K^{d\times d}$.

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Definition

A *spectrahedron in* K^n is any affine section of a set of the form

$$\mathcal{S}_A = \varphi_A^{-1}(K_{+}^{d\times d}) = \{x \in K^n : A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0\}$$

for some $d \in \mathbb{N}$, where $\varphi_A : K^n \to \mathcal{L} \subset K^{d\times d}$ is the map $\varphi_A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$, and $\mathcal{L} = A_0 + \operatorname{span}(A_1, \ldots, A_n)$.

Polyhedron implies Spectrahedron

Proposition

A polyhedron in K^n is a spectrahedron in K^n.

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Proof.

Basically, same idea as over $\mathbb{R}$: we can put the inequalities of valuation into some big diagonal matrix to get some linear matrix inequality. $\square$

Some spectrahedra are not polyhedra

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Example

The following spectrahedron is not a polyhedron:

$$\mathcal{S} = \left\{ x \in K : \begin{bmatrix} 0 & x + \pi^{-1} \\ x & 0 \end{bmatrix} \succeq 0 \right\}.$$

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Question

Are all spectrahedra of K diskoids?

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for some representatives $b_l \in O_K$ of the classes of $(O_K \setminus \{0\})/\pi O_K = \kappa^\times$. $\square$

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- Can we decide if a spectrahedron is non-empty?

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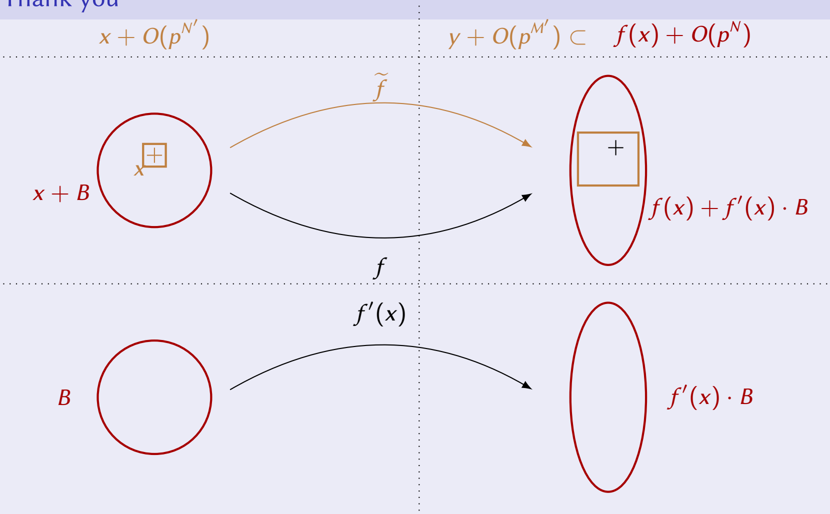
Future works

- Can we decide if a spectrahedron is non-empty?
- Can we give an explicit representation?

Thank you for your attention

Thank you



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