On OM Algorithm and Cluster Pictures II:
handling low precision

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Is there a good answer to **handle and clusterize all the roots** of the polynomials of the form $f + O(p^N)$?

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Discrete-valuation fields

Let K be a field with:

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Topology somehow in-between discrete and continuous mathematics.

Non-archimedean balls

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Definition (Gauss valuation)

Let $f = \sum_{i=0}^{d} a_i X^i \in K[X]$, we define the Gauss valuation of f as:

$$\mu_0(f) := \min_{0\leq i\leq d} v_K(a_i).$$

Remark

If $g = f + O(p^N)$ then

$$\mu_0(f-g) \geq N.$$

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David Brink: New light on Hensel's lemma, 2006

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$$\begin{array}{rccc} \Phi_\alpha : & \mathbb{R} \cup \{\infty\} & \longrightarrow & \mathbb{R} \cup \{\infty\} \\ & u & \mapsto & \displaystyle\sum_{l=1}^{d} \min\left(u, v_{\overline{K}}(\ \alpha - \alpha_l\ )\right). \end{array}$$

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Let $f = \prod_{k=1}^{d} (x - \alpha_k)$ and $f^* = \prod_{k=1}^{d} (x - \alpha_k^*)$. Their roots (in $\overline{K}$) can be ordered such that for all k,

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Proposition

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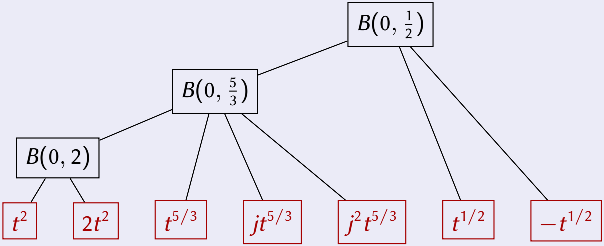

Two words on cluster pictures

What is a cluster picture?

Can be understood as a top-down representation of the Berkovich skeleton.

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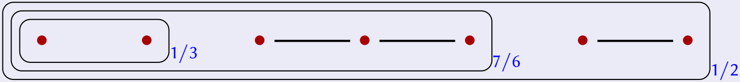

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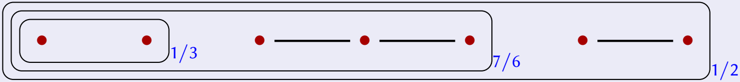

Applications of cluster pictures?

Semistable model, lattice of regular forms, regulator, etc... for curves in arithmetic geometry (see e.g. **Kunzweiler2021**; **Best2022**).

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OM - Berkovich correspondance

polynomial

$\downarrow$ *OM algorithms*

valuative tree — augmenting valuations using key polynomials φ and value λ

$\updownarrow$ *Ruth 2015, correspondance by the sup-norm*

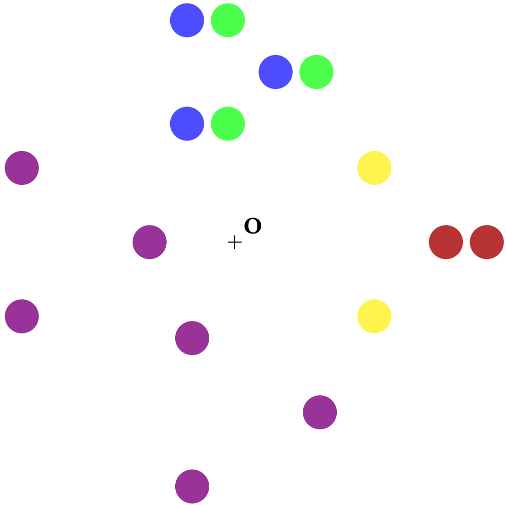
tree of diskoids — Corresponding discoid $D_K(\varphi; \lambda)$

$$\{x \in K : \; v_K(\varphi(x)) \geq \lambda\} = \cup_{i=1}^{d} B_{\overline{K}}(\alpha_i, \gamma)$$

$\downarrow$ *Kunzweiler 2021 + PVW 2025, only if there is no wild ramification*

Berkovich skeleton

The discoid point of view on the OM algorithm



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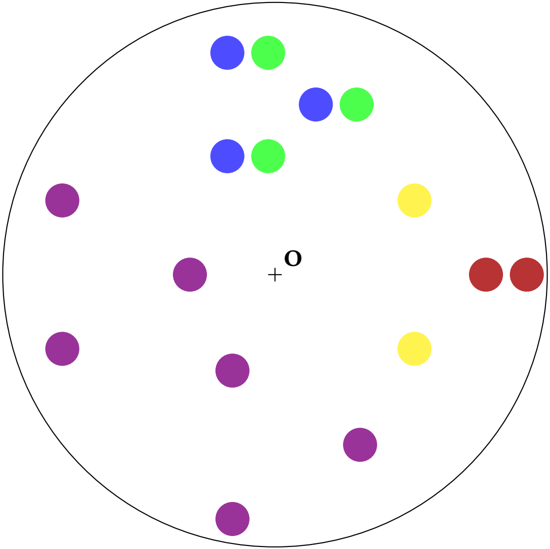

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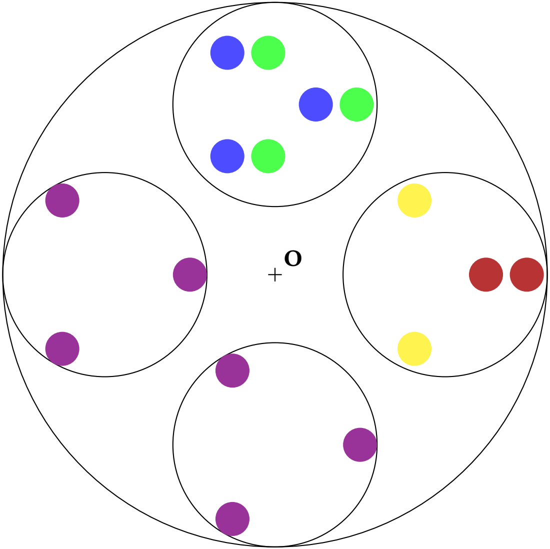

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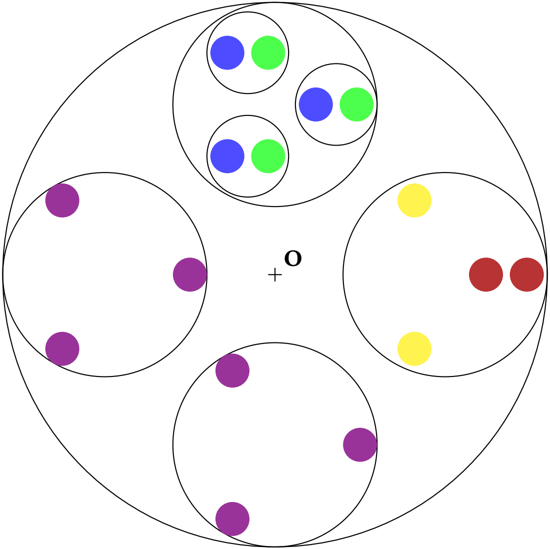

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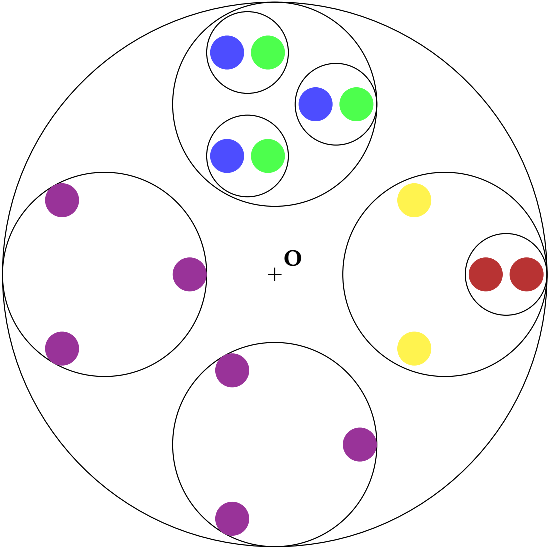

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We have adapted the OM algorithm to better handle small precision.

[1] Plus some univariate factorizations over κ of degree sum at most d and whose cost fits in the aimed bound if κ is finite.

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with the truncations $f_i + O(\pi^\sigma)$ distinct and irreducible.

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The Berkovich skeleton of the roots can still be recovered from the valuative tree of this OM algorithm.

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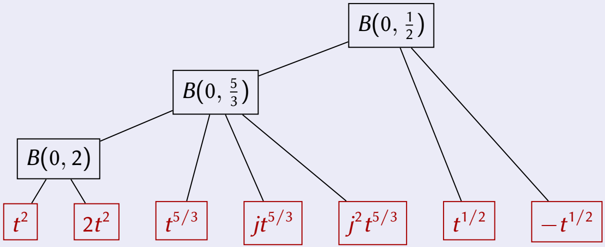

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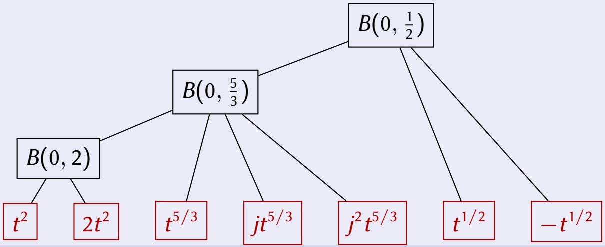

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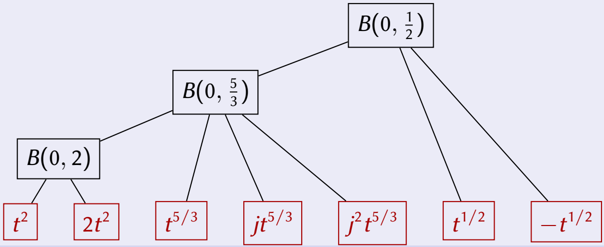

Matrix of the valuations of the differences of roots

$$MV_{f_5} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{3} & 0 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{3} & \frac{5}{3} & 0 & \frac{5}{3} & \frac{5}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & 2 & 0 \end{pmatrix}$$

Example: distance between roots to Brink's balls

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Thus

$$\Phi_{t^{\frac{1}{2}}} = 2\min\left(u, \frac{1}{2}\right) + 3\min\left(u, \frac{5}{3}\right) + \min(u, 2) + u.$$

Inverting Φ_α

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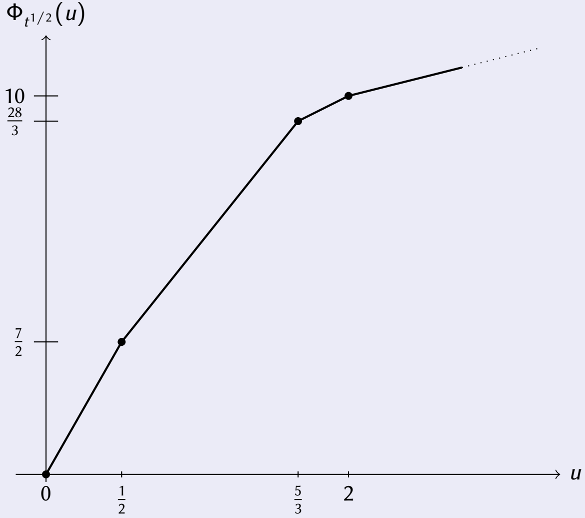

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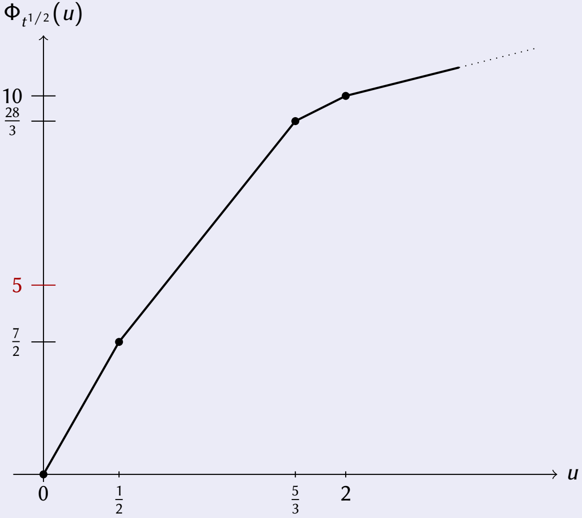

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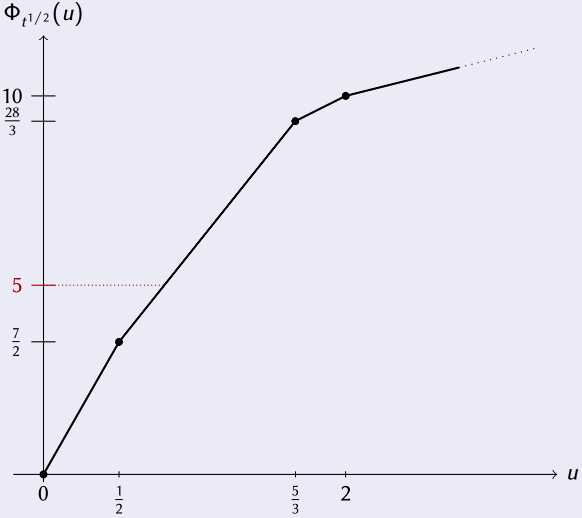

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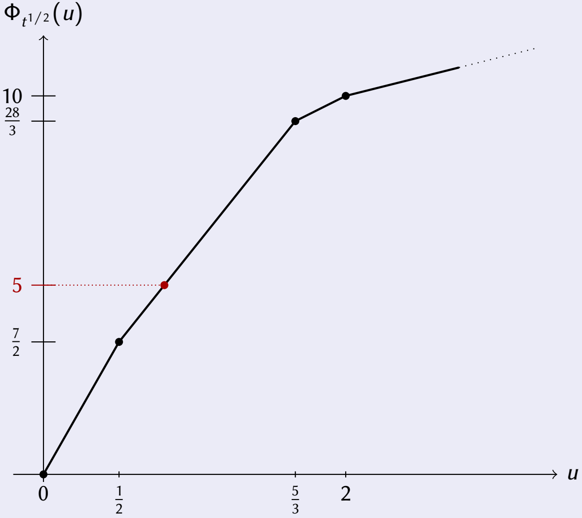

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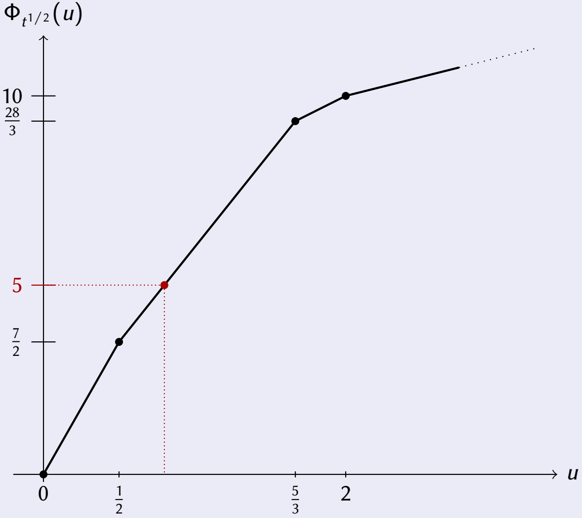

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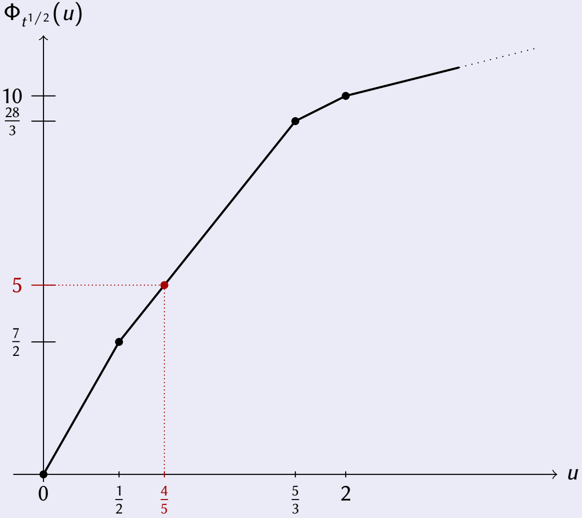

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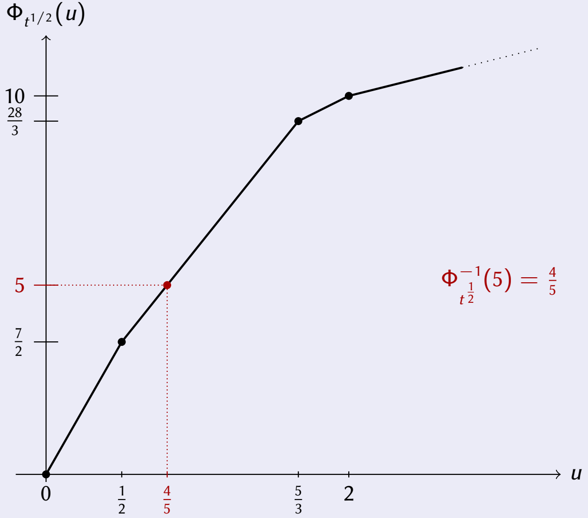

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Reorganization as a tree

Case of f_5 and precision 50: presenting all balls

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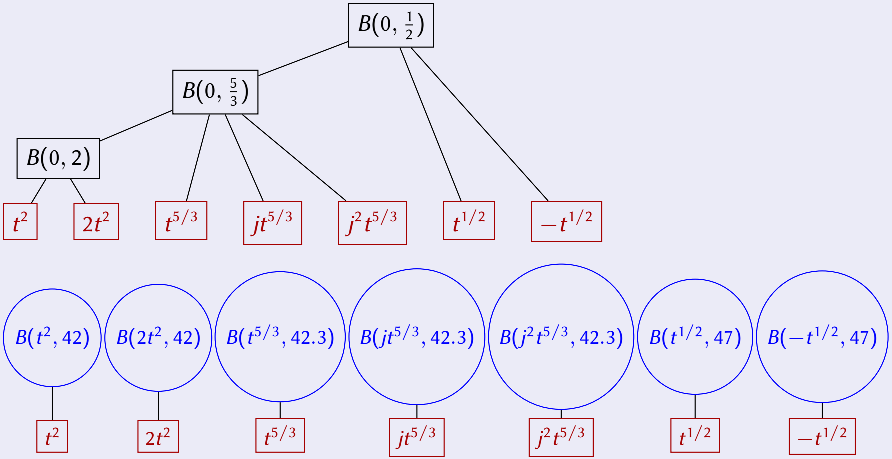
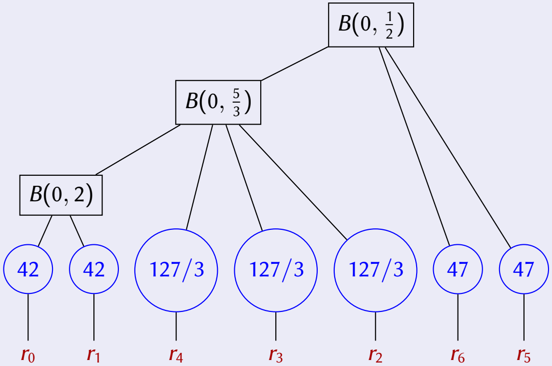


After computation: final form

Approximate Berkovich skeleton of $f_5 + O(\pi^{50})$ $(127/3 = 42.333\ldots)$



Final theorem

Proposition

Let $f \in K[x]$ be monic of degree d. Let $\sigma \in \mathbb{N}$.

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Let $f \in K[x]$ be monic of degree d. Let $\sigma \in \mathbb{N}$.
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- We can handle non-simple roots.

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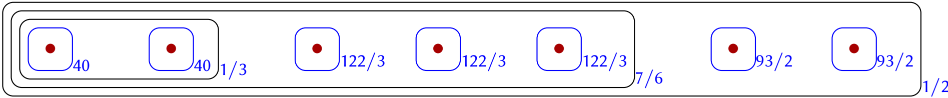

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Going back to the first example: $x^2 + O(t^{20})$

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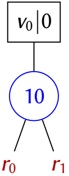

In black: Balls of given log-radii defined by diskoids defined by the following valuations:

- v_0: Gauss valuation induced by (t)-adic valuation

In blue: Balls coming from Brink's inequality.

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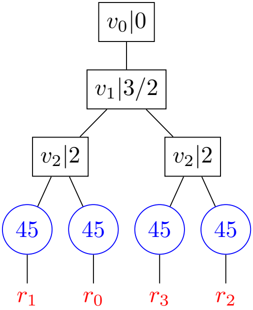
Case of $f_2 = (x^2 - t^3)^2 - xt^6 - t^7 \in k[[t]][x]$ at precision 50



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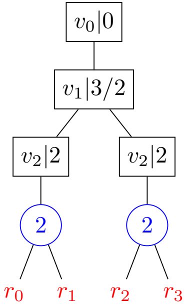

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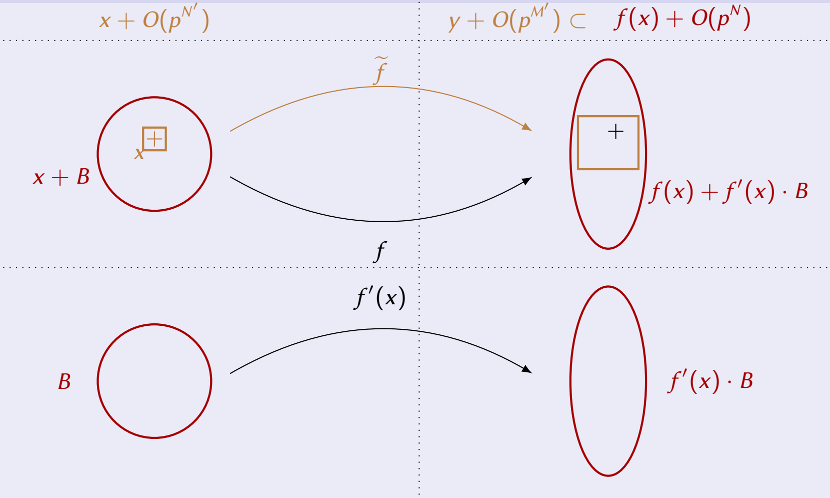
Future works

- More efficient implementation and integration to SageMath.
- Berkovich skeleton and cluster pictures in the case of wild ramification?

Thank you for your attention

Thank you



References

Cluster pictures, discoids, applications

- **Best2022**, *A user's guide to the local arithmetic of hyperelliptic curves*, Best, Alex J *et al.*, 2022.
- **Kunzweiler2021**, *Models of curves and integral differential forms*, Kunzweiler, Sabrina, PhD Thesis 2021.
- **Ruth2015**, *Models of curves and valuations*, Rüth, Julian, PhD Thesis 2015.

OM algorithm and related topics

- **HenselianFacto2024**: *Polynomial factorization over henselian fields*, Alberich-Carramiñana, Maria *et al.*, Journal of FOCM 2024.
- **vdHL2025**: *Plane curve germs and contact factorization* van der Hoeven, Joris and Lecerf, Grégoire in: AAECC 36.1 (2025),
- **PW2022**, *Local polynomial factorisation: improving the Montes algorithm*, Poteaux, Adrien and Weimann, Martin, ISSAC 2022.

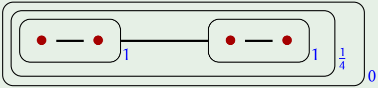
Correspondance between the two

- **PVW2025**, *On OM Algorithms and Cluster Pictures*, Poteaux, Adrien, **V., T.**, and Weimann, Martin, ISSAC 2025.

Bonus: wild (counter-)example

Example (**Kunzweiler2021**)

Let $\varphi_2 := x^4 - 2 \in \mathbb{Q}_2[x]$. Because $\mathrm{val}_2(2\sqrt[4]{2}) = \frac{5}{4}$, its cluster picture is:



However, φ_2 is irreducible and one can not get a longer chain of valuation than $[v_0, v_1(x) = \frac{1}{4}, v_2(x^4 - 2) = +\infty]$.
In other words, the two clusters with two roots and relative depth 1 can not be seen on the chain of valuations.

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Definition

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The Gauss valuation μ_0 is in $V(K[x])$.

Let $v \in V(K[x])$.

Definition

- The **valuation ring** of v is $O_v := \{f \in K[x] : \ v(f) \geq 0\}$.
- Its prime ideal is $O_v^+ := \{f \in K[x] : \ v(f) > 0\}$.
- its **residue ring** $\mathfrak{R}_v := O_v/O_v^+$ (always isomorphic to some $\kappa_v[\zeta]$ with κ_v a finite extension of κ)

Augmented valuation, inductive valuation

Definition

Let $v \in V(K[x])$, φ a **(key)** polynomial over v and $\lambda > v(\varphi)$.
We define the **augmented valuation**

$$v' = [v, v'(\varphi) = \lambda] \in V(K[x])$$

as follows:

$$\boxed{v'(f) := \min_i \left(v(f_i) + i\lambda\right)}$$

where the unique φ-expansion of f is $f = \sum_i f_i \varphi^i$.

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Definition

Starting from v_0, an **inductive valuation** is defined recursively as:

$$v_n = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$$

via $v_k := [v_{k-1}, v_k(\varphi_k) = \lambda_k]$ for $k = 1, \cdots, n$.

Bonus: Dissection by slope of Newton polygon: Classical case

Definition

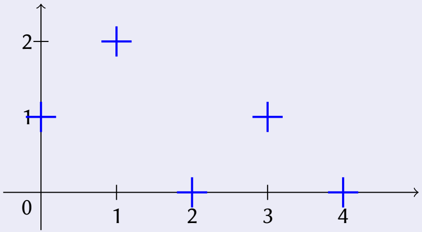
If $P = a_0 + a_1 X + \cdots + a_n X^n$, $NP(P)$ is the lower convex hull of the set

$$\{(i, \mathrm{val}(a_i)) \text{ for i in } [\![0, n]\!]\} .$$

Then P splits with one factor by slope of $NP(P)$ and these factors are relatively prime.

Example

$P = 3 + 9X + 7X^2 + 3X^3 + 2X^4$, over $\mathbb{Q}_3$



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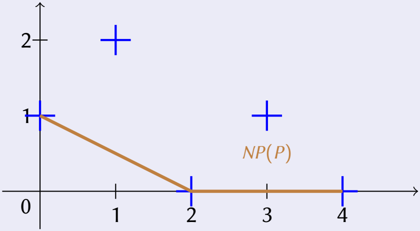

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Bonus: Dissection by slopes of Newton polygon: inductive valuation

Definition

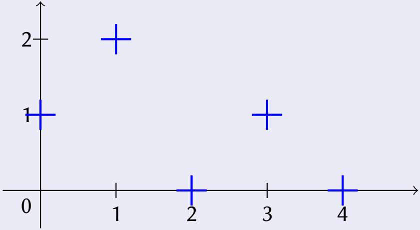
If μ is a valuation, $\varphi \in K[x]$ and $P = a_0 + a_1 X + \cdots + a_n X^n = b_0 + b_1 \varphi + \cdots + a_k \varphi^k$, $NP_{\mu,\varphi}(P)$ is the lower convex hull of the set

$$\{(i, \mu(b_i)) \text{ for i in } [\![0, n]\!]\}.$$

Then in the OM algorithm, applied on P, μ has one descendent by slope of NP_μ.

Example

$P = t + 5t^2\varphi + \varphi^2 + (7t + t^2)\varphi^3 + \varphi^4$, in $\mathbb{Q}((t))[x]$



Here v_0 is the Gaussian valuation.

$$\varphi = x^3 - t.$$

Bonus: Dissection by slopes of Newton polygon: inductive valuation

Definition

If μ is a valuation, $\varphi \in K[x]$ and $P = a_0 + a_1X + \cdots + a_nX^n = b_0 + b_1\varphi + \cdots + a_k\varphi^k$, $NP_{\mu,\varphi}(P)$ is the lower convex hull of the set

$$\{(i, \mu(b_i)) \text{ for i in } [\![0, n]\!]\}\,.$$

Then in the OM algorithm, applied on P, μ has one descendent by slope of NP_μ.

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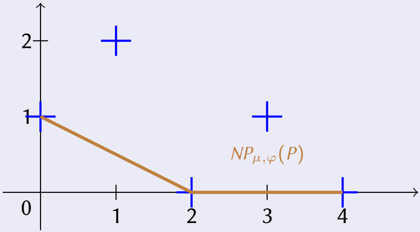

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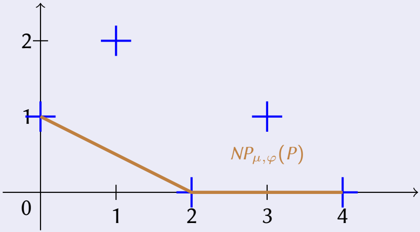

Here v_0 is the Gaussian valuation.

$$\varphi = x^3 - t.$$

Then v_0 splits into two valuations.

Bonus: Residual factorization

Theorem

If $f \in O_K[x]$ *is monic with*

$$f \bmod \pi = g \times h \in \kappa[x]$$

with g, h *relatively primes, then* f *splits in* $O_K[x]$ *with factors obtained from* g *and* h *by Hensel-Newton.*

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with g, h relatively primes, then f splits in $O_K[x]$ with factors obtained from g and h by Hensel-Newton.

Theorem

If $f \in O_K[x]$ is monic with reduction

$$\overline{f} = g \times h \in \mathfrak{R}_v$$

*with g, h relatively primes, then the augmentations of v should follow **two branches**: one for g and one for h.*

Bonus: Refinement of the valuation

Not splitting

If there is no dissection by residual polynomial nor by Newton polygon, then

$$\overline{f} = g^n \in \mathfrak{R}_v,$$

for some irreducible $g \in \mathfrak{R}_v$,

Bonus: Refinement of the valuation

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Refinement or irreducibility

- If $n = 1$: f is **irreducible**.

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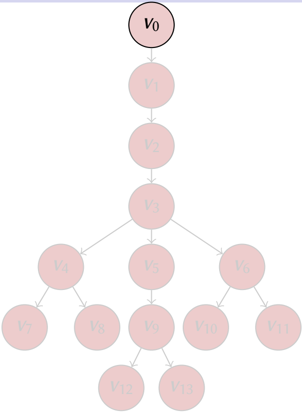
Refinement or irreducibility

- If $n = 1$: f is **irreducible**.
- Else, $n > 1$ and one **refines** v using an n-th root approximation of f.

Bonus: OM algorithm in practice: construction of the valuative tree

- $\varphi_1 = x^2 - t$,
- $\varphi_2 = \varphi_1^3 + 2\,t^4$,
- $\varphi_3 = (\varphi_2^5 + 4\,t^{27})\,(\varphi_2^3 - 2\,t^{13}) + t^{51}$,
- $f_3 = \varphi_3^2 - t^{82} + t^{83}$.
- f_3 has degree 96.



Inside the computation

Bonus: OM algorithm in practice: construction of the valuative tree

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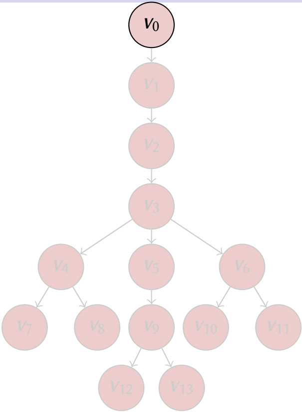

Inside the computation

- Residual polynomial of f_3 in $\mathfrak{R}_{v_0}$: x^{96}.
- $NP_{v_0,x}(f_3)$ has one slope: $-\frac{1}{2}$.
- Extension using $v_1(x) = \frac{1}{2}$.

Bonus: OM algorithm in practice: construction of the valuative tree

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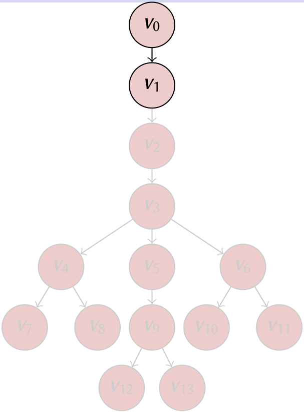

Inside the computation

- Residual polynomial of f_3 in $\mathfrak{R}_{v_1}$: $(x-1)^{48}$.
- $NP_{v_1, x^2 - t}(f_3)$ has one slope: $-\frac{4}{3}$.
- Extension with $v_2(x^2 - t) = \frac{4}{3}$, with $x^2 - t$ being a 48-th approximate root of f_3.

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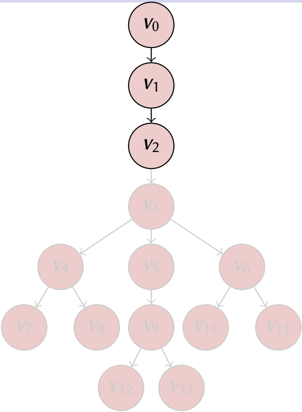

Inside the computation

- Residual polynomial of f_3 in $\mathfrak{R}_{v_2}$: $(x+2)^{16}$.
- $NP_{v_2, x^6 - 3tx^4 + 3t^2x^2 + 2t^4 - t^3}(f_3)$ has one slope: $-\frac{13}{3}$.
- Extension with $v_3(x^6 - 3tx^4 + 3t^2x^2 + 2t^4 - t^3) = \frac{13}{3}$.

Bonus: OM algorithm in practice: construction of the valuative tree

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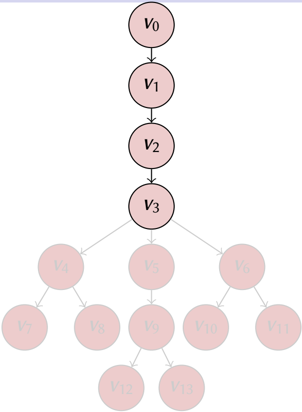

Inside the computation

- Residual polynomial of f_3 in $\mathfrak{R}_{v_3}$: $64(x+1)^2 x^{10} (x^2 - x + 1)^2$.
- Many extensions: v_4, v_5, v_6.

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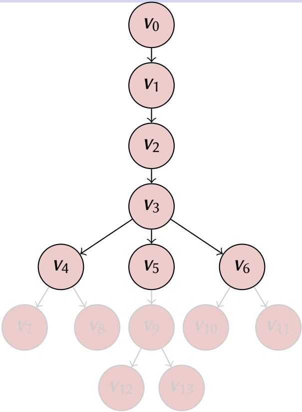

Inside the computation

More splitting and refinements.

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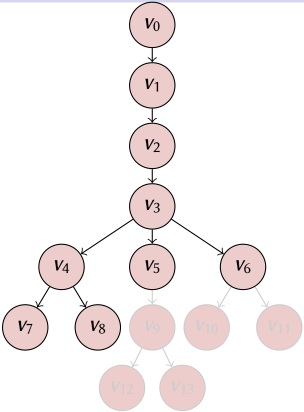

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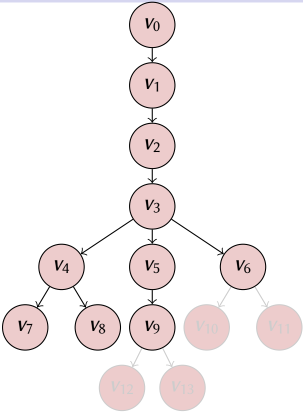

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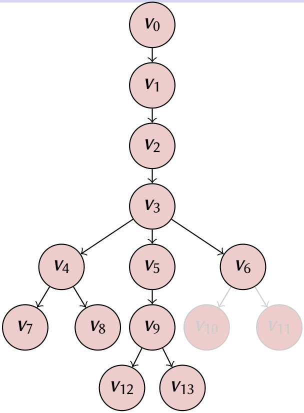

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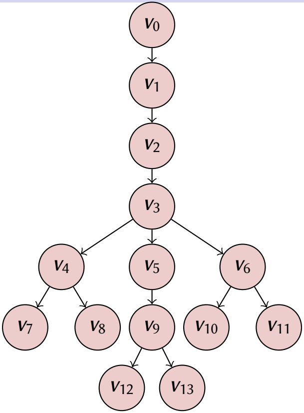
- f_3 has degree 96.
- f_3 has 6 irreducible factors



Inside the computation

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